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The Application of the DSGE-VAR Model to the Polish Macroeconomic Data

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Streszczenie

Celem niniejszej pracy jest estymacja modelu DSGE-VAR na podstawie sześciu zmiennych makroekonomicznych dla Polski. Zastosowany przy tym model DSGE jest modelem średniej wielkości, a jego specyfikacja w dużej mierze oparta jest na pracy Del Negro et al. (2007). Model DSGE-VAR umożliwia wykorzystanie zalet zarówno spójnych teoretycznie modeli strukturalnych, jak i modeli szeregów czasowych, charakteryzujących się znacznym stopniem dopasowania do danych empirycznych. Dodatkowo, estymacja bayesowska pozwala na uwzględnienie przekonań *a priori* dotyczących rozkładów parametrów modelu, co ma szczególne znaczenie w przypadku krótkich szeregów czasowych dla Polski. Ponadto uzyskane wyniki pozwalają na ocenę stopnia nieprawidłowości w specyfikacji modelu DSGE.

Słowa kluczowe: DSGE-VAR, modele DSGE, estymacja bayesowska, ocena modeli.

Abstract

The aim of this paper is to perform the estimation of a DSGE-VAR model using the six key macroeconomic variables for Poland, with the medium-scale DSGE model specified similarly to Del Negro et al. (2007). The DSGE-VAR approach enables to combine the advantages of the theoretically consistent structural models with those of the empirical ones, characterised by the substantial degree of data fit. Moreover, the Bayesian estimation provides a convenient framework to incorporate initial beliefs about the model parameters into the estimation procedure, which seems to be particularly advantageous in the case of rather short time series for Poland. Finally, the obtained estimates allow to assess the extend of the DSGE model misspecification.

Keywords: DSGE-VAR, DSGE models, Bayesian estimation, model evaluation.

Contents

List of Figures

List of Tables

Chapter 1

Introduction

For a long time, macroeconomic modelling has been faced with a tradeoff between theoretical coherence and data fit (see Sims 2008). On the one hand, micro-founded models, with understandable interpretation and clear implications, often did not explain the observed phenomena to a satisfactory degree. On the other hand, the large-scale macroeconometric models, able to track and forecast time series with a high precision, failed to provide a clarification of the economy structure or identification of shocks hitting the system.

Since the pioneering paper of Kydland and Prescott (1982), the dynamic stochastic general equilibrium (DSGE) models have been the subject of interest for both, practitioners and theorists. The ability of such structural models to detect the linkages between economic variables and to identify stochastic disturbances makes them a valuable tool in policy making. Moreover, the recent improvements of their empirical performance due to the work of Smets and Wouters (2003) have lead to increasing credibility of this class of models in monetary policy analysis. Nevertheless, the extend of data explanation provided by theoretical models will almost certainly be lower then in the case of atheoretical ones. Especially, the vector autoregressive (VAR) models, first introduced by Sims (1980), are renown for their profound power to capture the dynamic properties of the economic system, and therefore have been extensively applied by practitioners. The less restrictive specifications of empirical models are often a considerable advantage when compared with the DSGE models, which, even simple ones, impose very strong restrictions on actual time series. Unfortunately, VARs are often high-dimensional and densely parameterised, which poses estimation difficulties when dealing with scarcity of data. Precise inference, basing solely on the sample information, is in many cases impossible.

To deal with these problem, Bayesian inference is being applied in extending number of macroeconomic papers[1]. The Bayesian approach consists in updating former beliefs about parameters distribution with information coming from the data using the straightforward Bayes Theorem, allowing therefore for combining different sources of information. In Bayesian DSGE model estimation, prior distribution allows to incorporate knowledge about parameters, which are hardly identifiable from the aggregate data, wheres in

[1] According to Herbst (2010), in period 2005-2010 70% of all DSGE models, published in top eight economic journals, were estimated using Bayesian methods

VAR estimation, the role of priors is to reduce model dimensionality and variance of parameter estimates. Consequently, Bayesian methods make it possible to overcome the main obstacles of both modelling approaches.

A logical implication is therefore to combine the advantages of theoretical and empirical models through mixing them up in Bayesian fashion. Such a hybrid model, constructed by adding priors based on a DSGE model to a structural VAR, is likely to fit the data better then a DSGE model alone and, at the same time, to avoid the identification problems. This approach has become particularly popular since the influential paper of Del Negro and Schorfheide (2004), who reported that "the resulting [DSGE-VAR] model is competitive with standard benchmarks in terms of forecasting, and can be used for policy analysis". However, the empirical performance is not the only advantage of DSGE-VAR models: as presented in Del Negro et al. (2007), they can be successfully employed as a metric for assessing the degree of DSGE models misspecifications. Since one can control for the weight put on the DSGE model based prior, the estimated value of the tightness parameter delivers the answer to the question "how good is what you've got?"[2], i.e. indicates, how plausible the restrictions implied by the structural model are. No wonder therefore, that DSGE-VARs have been in centre of attention of both, academics and policy makers[3].

The aim of this paper is to construct and estimate a DSGE-VAR model for the Polish economy. As an underlying DSGE model, a modified version of the Smets and Wouters (2003) model (thereafter SW) will be used, which is renown for its excellent performance in terms of tracking and forecasting euro area time series. However, modelling the Polish economy in SW spirit does not necessarily hold the promise of achieving equally good outcomes as the cited authors. The degree of model misspecifications is likely to be higher owing to the specificity of the Polish transition economy on the one hand, and, on the other hand, the close-economy approach, which seems to be less suitable for the Polish economy than for the euro area. Nevertheless, since SW is nowadays considered as a benchmark for macro-modelling, and the presented DSGE-VAR approach allows for evaluating the degree of potential misspecifiations, it has been decided to follow SW approach.

The remainder of this paper is organised as follows. Chapter 2 explains the DSGE-VAR model and tackles with the issues related to it, i.e. identification problem and MCMC techniques. Chapter 3 presents the DSGE model and drafts the methods used to solve it. Chapter 4 reports on empirical analysis of the DSGE-VAR model. Chapter 5 deals with model evaluation. Finally, chapter 6 concludes and presents a brief outlook for further research.

[2]Which is the title of Del Negro and Schorfheide (2006) paper.

[3]Recent publications on this topic include application of DSGE-VAR model to macroeconomic data in the US (Adjemian et al. 2008), Japan (Watanabe 2009), New Zealand (Lees et al. 2007) or Singapore (Chow and McNelis 2010).

Chapter 2

DSGE-VAR model

The basic idea of a DSGE-VAR model is to incorporate additional information from a theoretical DSGE model to an econometrical VAR model. It is done within the Bayesian framework, where the prior assumption on the distribution of the VAR parameters is updated with the information coming from the data to obtain the posterior distribution of the VAR parameters. The key feature of DSGE-VAR approach consists in the way in which the prior distribution is constructed: it is derived from the observations generated from the DSGE model. Essentially, this approach leads to an estimation of the VAR based on a mixed sample of artificial and actual observations.

The starting point for the estimation is an unrestricted VAR model of order p. Let y_t be a $n \times 1$ vector of observed variables and θ be a vector of DSGE model parameters. A VAR(p) specification for y_t is given by:

$$y_t = \Phi_0 + \sum_{i=1}^{p} \Phi_i y_{t-i} + u_t, \tag{2.1}$$

where u_t is a vector of reduced-form innovations (one-step-ahead forecast errors), assumed to follow a serially independent multivariate normal distribution conditional on past information: $u_t \sim \mathcal{N}(0, \Sigma_u)$. Φ_0 stands for a $n \times 1$ vector of constants and Φ_i stands for $n \times n$ matrix of model coefficients, $\forall i = 1, \ldots, p$.

Following Schorfheide (2010), the DSGE-VAR approach can be understood as hierarchical hybrid model, which leads a nested structure of the form:

$$p(Y, \Phi, \Sigma_u, \theta) = p(Y|\Phi, \Sigma_u)p(\Phi, \Sigma_u|\theta)p(\theta), \tag{2.2}$$

where $p(\theta)$ stands for the prior for the DSGE parameters, $p(Y|\Phi, \Sigma_u)$ is the VAR likelihood function and $p(\Phi, \Sigma_u|\theta)$ denotes the prior for the VAR coefficients conditional on the DSGE model parameters. Therefore, by combining two sources of information, the model (generating the prior distribution) and the data (summarised by the likelihood function), the DSGE-VAR approach relaxes the tight theoretical restrictions of the DSGE model.

2.1 The Likelihood function

The model (2.1) can be expressed in a compact form:

$$Y = X\Phi + U, \tag{2.3}$$

where Y denotes the $T \times n$ matrix with rows y_t', X stands for the $T \times k$ matrix with rows $x_t' = [1, y_{t-1}', \ldots, y_{t-p}']$, where $k = 1 + np$, $\Phi = [\Phi_0, \Phi_1, \ldots, \Phi_p]'$ is the $k \times n$ matrix and U is the $T \times n$ matrix with rows u_t'. T stands for the sample size. The likelihood function of the VAR, conditional on observations $y_{1-p}, \ldots, y_0$, is given by:

$$p(Y|\Phi, \Sigma_u) \propto |\Sigma_u|^{-T/2} \exp\left\{-\frac{1}{2} tr[\Sigma_u^{-1}(Y - X\Phi)'(Y - X\Phi)]\right\}. \tag{2.4}$$

Following Del Negro and Schorfheide (2004), since the DSGE models do not have a finite-order VAR representation, the VAR(p) can be interpreted as an approximation to the VMA representation of the DSGE model. Obviously, the higher p, the better accuracy. However, increasing of the VAR order raises the number of parameters, which causes estimation difficulties. Therefore, to tackle the problem of too many parameters, the idea of "shrinkage estimators" has been developed. In the DSGE-VAR approach the DSGE model imposes tight restrictions on the VAR representation of y_t through a prior distribution obtained from the former model. According to An and Schorfheide (2006), the role of the prior for the VAR is reduction of the dimensionality of the econometric model - which is noticeably different from the role of prior in the DSGE model estimations.

2.2 Prior Distributions

To estimate the VAR parameters in a Bayesian fashion, the assumption concerning prior distribution for Φ and Σ_u is needed. In the DSGE-VAR model it is derived from the DSGE model and the joint prior for the VAR and for the DSGE model parameters is constructed. The method presented in this paper, based on Del Negro and Schorfheide (2004), Del Negro et al. (2007) and Adjemian et al. (2008), can be summarised as follows: choose a prior for the DSGE model parameters θ; for a given θ obtain a prior for the VAR coefficients with the use of the mapping from θ to Φ and Σ_u; multiply both priors to obtain a joint prior. Therefore, the joint prior has a hierarchical structure:

$$p(\Phi, \Sigma_u, \theta) = p(\Phi, \Sigma_u|\theta)p(\theta). \tag{2.5}$$

The issue is how to construct the mentioned mapping from the DSGE model parameters to the VAR parameters, i.e. how to find $p(\Phi, \Sigma_u|\theta)$. Following the previous literature, it can be achieved by augmenting the actual observations with the artificial data generated from the DSGE model and estimating the VAR model on the mixed sample of the actual and artificial data. The ratio of the latter over the former is given by the hyperparameter λ, which therefore can be understood as a weight of the prior relative to the sample or the prior tightness. The continuum of the DSGE-VAR(λ) models has an unrestricted VAR at the one extreme, for $\lambda = 0$, and VAR representation of the DSGE model at the other extreme, for $\lambda = \infty$.

In should be emphasised, that the hyperparameter λ plays a crucial role in the empirical performance of the DSGE-VAR(λ) model, since it determines the tightness of the prior for the VAR parameters and therefore controls the cross-coefficient restrictions imposed by the DSGE model the VAR. In order to allow its direct estimation as another parameter, following Adjemian et al. (2008), the prior distribution for λ is defined[1]. Because the initial beliefs concerning the optimal size of the generated sample suggest the presence of the DSGE model misspecifications, which are likely to manifest themselves in a low value of λ estimate, it is assumed that $\lambda \sim \mathcal{U}(0, 2)$. Moreover, it is assumed that λ is independent from θ, which will allow for factorisation of $p(\theta, \lambda)$ into $p(\theta)p(\lambda)$.

The idea to form the initial beliefs concerning the VAR parameters on the basis of the random sample obtained from the DSGE model is reasonable and sounds convincing. However, if the prior would actually be constructed on the basis of the randomly generated observations, it would cause the stochastic distribution of the prior[2]. Hence, the artificial sample moments should be replaced with their expected values:

$$\Gamma^*_{y,y}(\theta) = \mathbb{E}_\theta[y_t y_t'], \qquad \Gamma^*_{x,y}(\theta) = \mathbb{E}_\theta[x_t y_t'], \qquad \Gamma^*_{x,x}(\theta) = \mathbb{E}_\theta[x_t x_t'], \tag{2.6}$$

which can be computed analytically from the state-space representation of the DSGE model[3]. Formally, the entire procedure described above can be characterised as follows. The initial diffuse prior for the VAR parameters

$$p(\Phi, \Sigma_u) \propto |\Sigma_u|^{-(n+1)/2} \tag{2.7}$$

is updated with the information obtained from the generated sample according to the formula:

$$p(\Phi, \Sigma_u|\theta, \lambda) = \frac{p(\theta, \lambda|\Phi, \Sigma_u)p(\Phi, \Sigma_u)}{p(\theta, \lambda)} \propto p(\theta, \lambda|\Phi, \Sigma_u)p(\Phi, \Sigma_u). \tag{2.8}$$

Taking into consideration that the sample of artificial observations contains the information about the deep parameters θ and λ, the expression $p(\theta, \lambda|\Phi, \Sigma_u)$ in the above equation can be replaced by $p(Y^*(\theta, \lambda)|\Phi, \Sigma_u)$, which is the likelihood function of the generated sample, giving:

$$p(\Phi, \Sigma_u|\theta, \lambda) \propto p(Y^*(\theta, \lambda)|\Phi, \Sigma_u)p(\Phi, \Sigma_u). \tag{2.9}$$

Suppose that the actual observations are augmented with $T^* = \lambda T$ artificial ones, denoted Y^*, generated from the DSGE model conditional on parameter vector θ. Then likelihood of the artificial sample has

[1] This is a crucially different approach to the one developed by Del Negro and Schorfheide (2003), where a value of λ was chosen over a finite grid to maximise the marginal data density of the DSGE-VAR(λ).

[2] In repeated application of the procedure the prior would display a stochastic variance.

[3] It can be done due to the assumed weak stationarity of y_t.

the form:

$$p(Y^*(\theta,\lambda)|\Phi,\Sigma_u) \propto |\Sigma_u|^{-\lambda T/2} \exp\left\{-\frac{1}{2}tr[\Sigma_u^{-1}(Y^* - X^*\Phi)'(Y^* - X^*\Phi)]\right\}. \tag{2.10}$$

Putting together (2.6), (2.7), (2.9) and (2.10) yields the formula for the VAR parameters prior, given the DSGE model parameters:

$$p(\Phi,\Sigma_u|\theta,\lambda) = c^{-1}(\theta)|\Sigma_u|^{-(\lambda T+n+1)/2} \exp\left\{-\frac{1}{2}tr[\lambda T\Sigma_u^{-1}(\Gamma^*_{yy}(\theta) - \Phi'\Gamma^*_{xy}(\theta) - \Gamma^*_{yx}(\theta)\Phi + \Phi'\Gamma^*_{xx}(\theta)\Phi)]\right\}, \tag{2.11}$$

where $c(\theta)$ is a normalising constant since the density has to integrate to one. Moreover, to ensure that (2.11) is nondegenerate and proper, $\Gamma^*_{xx}(\theta)$ should be invertible and $\lambda \geq k + n$. From (2.11) follows that the prior density for the VAR coefficients, conditional on the DSGE model parameters, is of the Inverse-Wishart-Normal form, which is a conjugate prior for the multivariate normal likelihood function with mean and covariance taken as parameters - which is the case for (2.4). Thus,

$$\begin{aligned} \Sigma_u|\theta,\lambda &\propto \mathcal{IW}\left(T^*\Sigma_u^*(\theta), T^* - k, n\right), \\ \Phi|\Sigma_u,\theta,\lambda &\propto \mathcal{N}\left(\Phi^*(\theta), \Sigma_u \otimes (T^*\Gamma^*_{xx}(\theta))^{-1}\right), \end{aligned} \tag{2.12}$$

where:

$$\begin{aligned} \Phi^*(\theta) &= \Gamma^{*-1}_{xx}(\theta)\Gamma^*_{xy}(\theta), \\ \Sigma^*_u(\theta) &= \Gamma^*_{yy}(\theta) - \Gamma^*_{yx}(\theta)\Gamma^{*-1}_{xx}(\theta)\Gamma^*_{xy}(\theta). \end{aligned} \tag{2.13}$$

The functions $\Phi^*(\theta)$ and $\Sigma^*_u(\theta)$ are called restriction functions and denote the VAR approximation of the DSGE model, at which the prior (2.11) is centred. Essentially, they indicate the mapping from the DSGE to VAR parameters. The VAR with the coefficient matrix $\Phi^*(\theta)$ and forecast error covariance matrix $\Sigma^*_u(\theta)$ minimises the one-step-ahead quadratic forecast loss among the p-th order VAR models - for a given DSGE parameters θ.

Finally, defining the prior distributions for the DSGE model structural parameters , $p(\theta)$ and $p(\lambda)$, allows to specify the prior distribution of the DSGE-VAR model in the following nested way:

$$p(\Phi,\Sigma_u,\theta,\lambda) = p(\Phi,\Sigma_u|\theta,\lambda)p(\theta)p(\lambda). \tag{2.14}$$

2.3 Posterior Distribution

The Bayes theorem implies that the posterior density is given by the product of the likelihood function and the prior density, which combined with the hierarchical structure of the model leads to the following formula for the posterior:

$$p(\Phi,\Sigma_u,\theta,\lambda|Y) \propto p(Y|\Phi,\Sigma_u)p(\Phi,\Sigma_u|\theta,\lambda)p(\theta)p(\lambda). \tag{2.15}$$

However, one can make use of the fact that (2.11) is a conjugate prior for (2.4) - which leads to the posterior for the VAR parameters given the DSGE model parameters of the known, Inverse-Wishart-

Normal form - and factorise (2.15) into:

$$p(\Phi, \Sigma_u, \theta, \lambda|Y) = p(\Phi, \Sigma_u|\theta, \lambda, Y)p(\theta, \lambda|Y). \tag{2.16}$$

Then the first term of the right-hand-side product is given by:

$$\begin{aligned} \Sigma_u|\theta, \lambda, Y &\propto \mathcal{IW}\left((1+\lambda)T\tilde{\Sigma}_u(\theta,\lambda), (1+\lambda)T-k, n\right), \\ \Phi|\Sigma_u, \theta, \lambda, Y &\propto \mathcal{N}\left(\tilde{\Phi}(\theta,\lambda), \Sigma_u \otimes \frac{1}{T}(\Gamma_{xx} + \lambda\Gamma^*_{xx}(\theta,\lambda))^{-1}\right), \end{aligned} \tag{2.17}$$

where $\tilde{\Phi}$ and $\tilde{\Sigma}_u$ are the maximum-likelihood estimates of Φ and Σ_u, respectively, based on the mixed sample:

$$\begin{aligned} \tilde{\Phi}(\theta,\lambda) &= (\Gamma_{xx} + \lambda\Gamma^*_{xx}(\theta))^{-1}(\Gamma_{xy} + \lambda\Gamma^*_{xy}), \\ \tilde{\Sigma}_u(\theta,\lambda) &= \frac{1}{1+\lambda}\left[(\Gamma_{yy} + \lambda\Gamma^*_{yy}(\theta)) - (\Gamma_{yx} + \lambda\Gamma^*_{yx}(\theta))(\Gamma_{xx} + \lambda\Gamma^*_{xx}(\theta))^{-1}(\Gamma_{xy} + \lambda\Gamma^*_{xy}(\theta))\right]. \end{aligned} \tag{2.18}$$

In coherence with the notation used in (2.6), Γ_{yy}, Γ_{yx} and Γ_{xx} stand for the moments of the actual sample.

Obviously, the higher λ, the larger weight of the prior, the closer the posterior mean of the VAR parameters conditional on θ is to the restriction functions Φ^* and Σ^*_u, i.e. $\tilde{\Phi}(\theta) \xrightarrow{\lambda\to\infty} \Phi^*(\theta)$ and $\tilde{\Sigma}_u(\theta) \xrightarrow{\lambda\to\infty} 0$. Thus, with the increase of the artificial sample size the VAR parameter estimates stay closer to the restrictions implied by the DSGE model. On the other hand, $\lambda = 0$ means taking the OLS estimate of the Φ as the posterior mean of Φ.

The second expression in (2.16), the posterior density of θ, is more cumbersome and in order to sample from it numerical methods have to be used. It can be expressed as:

$$p(\theta, \lambda|Y) \propto p(Y|\theta, \lambda)p(\theta)p(\lambda), \tag{2.19}$$

i.e. proportion to the product of the likelihood function $p(Y|\theta, \lambda)$ and the prior densities of θ and λ. The likelihood of θ and λ can be written as:

$$p(Y|\theta, \lambda) = \int p(Y|\Phi, \Sigma_u)p(\Phi, \Sigma_u|\theta, \lambda)d(\Phi, \Sigma_u). \tag{2.20}$$

or

$$p(Y|\theta, \lambda) = \frac{p(Y|\Phi, \Sigma_u)p(\Phi, \Sigma_u|\theta, \lambda)}{p(\Phi, \Sigma_u|Y)} \tag{2.21}$$

where the right-hand term in (2.21) can be expressed by a closed-form formula[4]. The prior density for θ is generated from the well-known densities and will be explained in further part of the paper.

[4]Del Negro and Schorfheide (2004)), page 34.

2.4 Identification Problem

One of the main tasks of the macroeconometrical modelling is to find the linkages between variables and therefore to detect the impact of various shocks on the macroeconomic variables. However, the estimation of the reduced-form VAR, as in (2.1), does not deliver information concerning the structure of the disturbances[5]. Consequently, to identify dynamic responses of separate variables to a given shock, the mapping between structural shocks ε_t and the reduced-form forecast errors u_t needs to be constructed, i.e. on the basis of estimated reduced-form VAR one needs to identify the underlying orthogonal shocks ε_t. Yet, this is a difficult undertaking and "even though several procedures have been proposed in the literature, shock identification remains a highly controversial issue" (Liu and Theodoridis (2010)).

In practice, the recursive method described by Sims (1980) is being commonly used. The relationship between u_t and ε_t is characterised as follows:

$$u_t = \Sigma_{tr}\Omega\varepsilon_t, \tag{2.22}$$

where Ω is an orthonormal matrix (called rotation matrix) and Σ_{tr} is the Cholesky decomposition of Σ_u. The matrix Ω provides the link between reduced-from and structural shock and therefore plays a key role in the identification procedure: in order to obtain impulse response functions (IRF) to unanticipated disturbances, it is necessary to define Ω. However, Ω is not identifiable from the data, i.e. the likelihood function of the VAR depends only on $\Sigma_u = \Sigma_{tr}\Sigma_u'$ and is invariant to Ω[6], which leads to the identification problem. The identification strategy developed by Del Negro and Schorfheide (2004), an natural approach in the DSGE-VAR framework, is based on the DSGE model. Ω is derived from the state-space representation of the DSGE model and should ensure that the IRFs from the DSGE model match those from the DSGE-VAR model - in the case of no misspecification.

The starting point for the procedure is to note that the DSGE model, as a structural one, is identified, which means that for each θ there exists a unique matrix $A(\theta)$ describing the contemporaneous effect of ε_t on y_t. $A(\theta)$ is obtained from the state-space representation and can be factorised with the QR decomposition as follows:

$$\left(\frac{\partial y_t}{\partial \varepsilon_t'}\right)_{DSGE} = A(\theta) = \Sigma_{tr}^*(\theta)\Omega^*(\theta), \tag{2.23}$$

where $\Sigma_{tr}^*(\theta)$ is a lower triangular matrix and $\Omega^*(\theta)$ is an orthonormal matrix. On the other hand, the initial impact of ε_t on the endogenous variables in the VAR is given by:

$$\left(\frac{\partial y_t}{\partial \varepsilon_t'}\right)_{VAR} = \Sigma_{tr}\Omega. \tag{2.24}$$

The VAR identification involves replacing Ω in (2.24) with $\Omega^*(\theta)$ from (2.23), while the Cholesky decomposition of Σ_u is maintained. The implementation of the procedure takes place within the MCMC algorithm (which samples Φ, Σ_u and θ from (2.16)) and consists of the following steps:

[5]This is because the errors in the u_t are correlated with each other.

[6]Since $\Sigma_{tr}\Omega\Omega'\Sigma_{tr}' = \Sigma_{tr}\Sigma_{tr}' = \Sigma_u$.

1. Construct a MA representation of y_t in terms of u_t using Φ.

2. Compute Σ_{tr}.

3. Calculate $\Omega = \Omega^*(\theta)$ and construct a MA representation of y_t in terms of ε_t.

As stated in Del Negro et al. (2007), "combining $\Omega^*(\theta)$ with the reduced-form VAR approximation of the DSGE model results in a structural VAR that mimics the impulse response dynamics of the DSGE model". Therefore, using of $\Omega^*(\theta)$ facilitates turning of the reduced-form DSGE-VAR into the identified DSGE-VAR.

In spite of coherence of the presented method, it is not free from substantial inconveniences. Firstly, the DSGE model needs to be fully stochastically specified to allow for employing the rotation matrix from it to identify the VAR. Secondly, even though it is possible to control for the prior weight of the artificial observations, one cannot control for the prior weight of the implied dynamics of the DSGE model. Thirdly, according to Sims (2008), the same identification problem as in the case of the standard VAR arises in the DSGE-VAR approach, which causes difficulties in putting DSGE-VARs into practise. The key issue is that the VAR representation to the DSGE model is only an approximation and therefore using the DSGE rotation matrix does not yield the same covariance matrix as the DSGE-VAR[7].

2.5 MCMC Algorithm

The Bayesian inference about the DSGE-VAR model parameters causes the necessity of dealing with multidimensional and highly nonlinear probability distributions, which usually do not have a closed-form representation (as in the case of $p(\theta, \lambda|Y)$) and thus have to be approximated numerically, i.e. by random draws from them. Since the size of the θ is usually large and initially little is know about the posterior it is difficult to generate independent draws from the posterior. To tackle this problem, the posterior is simulated with the Markov Chain Monte Carlo (MCMC) techniques, where a Markovian sequence $\{(\theta, \lambda)_i\}_{i=1}^N$ is constructed, which, by the ergodic theorem, converges to the desired posterior density as N becomes large. The most popular MCMC technique is the Metropolis-Hastings (MH) algorithm, which can be described as follows[8]:

1. Choose the initial value θ_0 and set $n = 1$.

2. While $n < N$ do:

   2.1. Draw a proposal θ^* from $q(\cdot|\theta_{n-1})$.

   2.2. Calculate an acceptance ratio:

$$\alpha := \frac{p(\theta^*|Y)}{p(\theta_{n-1}|Y)} \frac{q(\theta_{n-1}|\theta^*)}{q(\theta^*|\theta_{n-1})}.$$

[7] Liu and K. (2010), page 7.

[8] For simplicity, in the following pseudocode λ is assumed to be a part of the vector of parameters θ.

2.3. $\theta_n := \begin{cases} \theta^* & \text{with probability } \min\{\alpha, 1\}, \\ \theta_{n-1} & \text{with probability } 1 - \min\{\alpha, 1\}. \end{cases}$

The distribution $q(\cdot|\theta_n)$ in (2.1.) is the proposal distribution, and one samples from it in order to avoid drawing from the complex distribution of interest, $p(\theta_n|Y)$. It is also referred to as transition kernel, since it generates an underlying Markov Chain. Therefore its support must contain the support of the posterior. The first term on the right hand side in (2.2.) is the likelihood ratio between the proposed sample θ^* and the previous sample θ_{n-1}. The second term stands for the ratio of the proposal density in two directions, from θ_{n-1} to θ^* and from θ^* to θ_{n-1}, and is equal to 1 for symmetric proposal densities.

The general idea of the MH algorithm is straightforward, however, two crucial issues arise when it comes to its practical implementation. Firstly, how to choose the starting value θ_0. Secondly, what the exact specification of the proposal distribution $q(\cdot|\theta)$ should be like. The former question is usually tackled with the numerical optimisation of the log posterior density and then setting θ_0 equal to the computed mode. The latter case is less routine, because there is no universal solution which works out in each application. On the one hand, transition kernel ought to have a simple form to facilitate drawing from it; on the other hand, it should approximate the posterior well - at least locally. For the DSGE models, the most common MH algorithm is the random walk Metropolis-Hastings (RWMH) algorithm, first proposed by Shorfheide (2000)[9]. The proposal takes here the form of:

$$\theta^* \sim \theta_{n-1} + \varepsilon, \qquad \text{where} \quad \varepsilon \sim \mathcal{N}(0, -cH^{-1}),$$

and H is the Hessian of the log posterior evaluated at the mode and c is an adjustment coefficient.

FollowingAn and Schorfheide (2007) and Adjemian et al. (2008), the complete MCMC algorithm for the DSGE-VAR estimation can be presented as follows:

1. Use the RWMH algorithm to generate draws $(\theta, \lambda)_n$ from the posterior distribution $p(\theta, \lambda|Y)$.
2. For each draw $(\theta, \lambda)_n$:
    1. sample Φ_n and Σ_n from $p(\Phi, \Sigma_u|\theta, \lambda, Y)$,
    2. compute Ω_n.

The practical implementation of the whole procedure described above can be carried out using the Dynare packet (Adjemian et al., 2011).

[9]According to Herbst (2010), the RWMH algorithm is used in 95% of papers where Bayesian interference is applied.

Chapter 3

DSGE Model

3.1 The Model

The model presented in this paper is based mainly on work of Del Negro et al. (2007), which is in turn a modified version of the model developed by Smets and Wouters (2003). However, several simplifications to the standard framework have been introduced: mark-up shocks have been removed, price indexation has been changed to purely backward-looking, the utilisation of capital has been omitted. Moreover, the government fiscal policy function has been replaced with an exogenous spending shock.

3.1.1 Household Sector

There is a continuum of households in the economy, indexed by $j \in [0,1]$, each of which supplies differentiated type of labour. Besides that fact, the households are homogenous, i.e. they have the same preferences and endowments.

The j-th household maximises its intertemporal expected utility:

$$\mathbb{E}_t \sum_{s=0}^{\infty} \beta^s U_{t+s}(j),$$

where $U_t(j)$ is an instantaneous utility function of the form:

$$U_t(j) = \varepsilon_{d,t} \left[\log(C_t(j) - hC_{t-1}(j)) - \frac{\varepsilon_{l,t}}{1+\varphi} L_t(j)^{1+\varphi} + \varepsilon_{m,t} \log\left(\frac{M_t(j)}{Z_t P_t}\right) \right].$$

Therefore, per period utility is a functions of consumption bundle $C_t(j)$, labour effort $L_t(j)$ and real money holdings $\frac{M_t(j)}{Z_t P_t}$[1]. $\mathbb{E}_t$ denotes expectation operator conditional on information available at time t; β is the discount factor; h represents the external degree of habit persistence; φ is the inverse of the Frisch elasticity of labour supply (elasticity of work effort with respect to the real wage).

[1] In order to make real money demand stationary, real money holdings are deflated by the stochastic trend growth of the economy.

There are three shock shifting the period utility. First, $\varepsilon_{d,t}$, affects the intertemporal preferences, i.e. the household's willingness to substitute over time (it is a demand shock, which scales the overall period utility). Second, $\varepsilon_{l,t}$, represents the labour supply shock and captures the movements in the observed wedge in the first order condition relating consumption and labour. Third, $\varepsilon_{m,t}$, is a shock to preferences to the money holdings. For simplicity, it is postulated that all these shocks follow an AR(1) processes in logs:

$$\begin{aligned}
\log \varepsilon_{d,t} &= \rho_d \log \varepsilon_{d,t-1} + \sigma_d \mu_{d,t}, && \text{where} \quad \mu_{d,t} \sim \mathcal{N}(0,1), \\
\log \varepsilon_{l,t} &= \rho_l \log \varepsilon_{l,t-1} + \sigma_l \mu_{l,t}, && \text{where} \quad \mu_{l,t} \sim \mathcal{N}(0,1), \\
\log \varepsilon_{m,t} &= \rho_m \log \varepsilon_{m,t-1} + \sigma_m \mu_{m,t}, && \text{where} \quad \mu_{m,t} \sim \mathcal{N}(0,1).
\end{aligned}$$

The household's intertemporal nominal budget constraint is given by:

$$\begin{aligned}
&P_t C_t(j) + P_t I_t(j) + M_t(j) + B_t(j) = \\
&= R_{t-1} B_{t-1}(j) + M_{t-1}(j) + \Pi_t + W_t L_t + R_t^k K_t(j),
\end{aligned} \tag{3.1}$$

where $I_t(j)$ denotes investment, $M_t(j)$ - nominal money holdings, $B_t(j)$ - holdings of government bonds, Π_t - dividend from ownership of the imperfect competitive intermediate firms, $K_t(j)$ - physical capital rented by the household to the firms. P_t stands for the price of the composite good (which can be used either for consumption or for investment), $W_t(j)$ - for nominal wage earned by the j-th household, R_t - for the gross nominal interest rate on government bonds and R_t^k - for the rate of return on rented capital.

Owing to the fact that households are exclusive owners of the capital in the economy, they make investment decisions affecting the size of the capital stock. The capital is being accumulated according to:

$$K_{t+1} = (1-\delta) K_t + \varepsilon_{i,t} F(I_t, I_{t-1}).$$

where δ is the depreciation rate and $F(I_t, I_{t-1})$ is a function, which turns investment into capital. Following Christiano et al. (2005), it is assumed that the specification for F is given by:

$$F(I_t, I_{t-1}) = \left(1 - S\left(\frac{I_t}{I_{t-1}}\right)\right) I_t,$$

where S is an investment effectiveness function, representing the cost of investment adjusting. It equals zero in the steady state, where the growth rate of investment is constant and equals e^{γ}. Moreover, it is assumed, that the first derivative of S is equal to zero in the equilibrium, while its second derivative is positive for all arguments. It implies that the adjustment costs depend only on the curvature of the effectiveness function. Formally:

$$S(e^{\gamma}) = S'(e^{\gamma}) = 0, \qquad S''(\cdot) > 0 \qquad \text{and} \qquad S''(e^{\gamma}) \equiv S''.$$

Capital accumulation is shifted by the investment-specific technological progress, $\varepsilon_{i,t}$, which follows the exogenous process:

$$\log \varepsilon_{i,t} = \rho_i \log \varepsilon_{i,t-1} + \sigma_i \mu_{i,t}, \quad \text{where} \quad \mu_{i,t} \sim \mathcal{N}(0,1).$$

This progress alters the rate of transformation between consumption and investment goods and can be interpreted as a stochastic disturbance to the price of investment relative to consumption.

It is assumed that there exist a complete set of Arrow-Debreu securities, contingent on idiosyncratic and aggregate states of nature. However, they are not explicitly included in the budget constraint[2]. Such an assumption implies that the Lagrange multipliers associated with the budget constraint (3.1) must be equal for all households across all periods and all states of nature. Thus, in the equilibrium, households decisions regarding $C_t(j)$, $M_t(j)$ and $I_t(j)$ will be identical. Therefore, the presence of the complete markets allows for omitting the j index. However, the choice of $L_t(j)$ will differ across the households due to wage rigidities, which will be introduced in the next section.

Moreover, to rule out the Ponzi schemes, the usual transversality condition on asset accumulation is assumed.

3.1.2 Labour Market

The labour supply is individualised and household-specific, therefore there is a need to aggregate it into composite labour services, which could be used by the intermediate good producers. This is the task of so called “labour packers”, perfectly competitive firms, which hire labour from the households, combine it into the aggregate L_t and resell it to the intermediate good producers. Aggregation takes place according to the Dixit-Stiglitz formula:

$$L_t = \left[\int_0^1 L_t(j)^{\frac{\phi_w - 1}{\phi_w}} dj \right]^{\frac{\phi_w}{\phi_w - 1}},$$

where ϕ_w is substitution elasticity among labour varieties. Given the global labour demand L_t, from labour packers' first-order conditions follows that the demand for the j-th type of labour services is equal to:

$$L_t(j) = \left[\frac{W_t(j)}{W_t} \right]^{-\phi_w} L_t. \tag{3.2}$$

Taking into consideration zero-profit condition of the labour packers, one obtains the price of the aggregated labour services:

$$W_t = \left[\int_0^1 W_t(j)^{1-\phi_w} dj \right]^{\frac{1}{1-\phi_w}}$$

Wage setting mechanism

Since each household is a monopolistic supplier of its own labour services, it posses some degree of monopolistic power on the labour market. Therefore, each household can set its labour price as in the case of imperfect competition. However, household's wage setting is subject to nominal rigidities, which are assumed to follow the Calvo mechanism (see Calvo, 1983). It is assumed that each period only a fraction $1 - \theta_w$ of households can readjust their wages. The wages of the remaining households are

[2]See Del Negro et al. (2004).

automatically adjusted by partial indexation to the previous period inflation π_{t-1}, where $\pi_t = \frac{P_t}{P_{t-1}}$:

$$W_t(i) = (\pi_{t-1})^{\kappa_w} W_{t-1}(i),$$

where $\kappa_w \in [0, 1]$ is a degree of indexation to past prices during the period in which a household is not allowed to renegotiate its wage.

A household that is allowed to renegotiate its wage chooses a wage $\tilde{W}_t(j)$ maximising its utility in all states of nature, in which this wage will hold at the chosen value[3]:

$$\max_{\tilde{W}_t(j)} \mathbb{E}_t \sum_{s=0}^{\infty} (\theta_w \beta)^s \varepsilon_{d,t} \left(-\frac{\varepsilon_{l,t}}{1+\varphi} L_t(j)^{1+\varphi} \right)$$

subject to:

$$\begin{aligned} \forall s \quad & \text{eq.(3.1)}, \\ & \text{eq.(3.2)}, \\ & W_{t+s}(j) = \begin{cases} \tilde{W}_t(j) & \text{for} \quad s = 0, \\ \tilde{W}_t(j) \prod_{k=1}^{s} (\pi_{t+k-1})^{\kappa_w} = \tilde{W}_t(j) \left(\frac{P_{t+s-1}}{P_{t-1}} \right)^{\kappa_w} & \text{for} \quad s > 0. \end{cases} \end{aligned}$$

Because of complete market structure the problem is symmetric, which means that all households able to reoptimise will set their wages at the same level. Thus, again, the index j can be omitted and $\tilde{W}_t(j) = \tilde{W}_t \quad \forall j$. Then the evolution of the aggregate wage index is given by the formula:

$$W_t = \left[(1-\theta_w) \tilde{W}_t^{1-\phi_w} + \theta_w (\pi_{t-1}^{\kappa_w} W_{t-1})^{1-\phi_w} \right]^{\frac{1}{1-\phi_w}}.$$

3.1.3 Final Goods Producers

The final good in the economy, Y_t, is produced by perfectly competitive firms, called the "aggregators". They buy intermediate goods on the market, combine them into a composite and resell to the households. The bundle is made from continuum of intermediate goods, indexed $i \in [0, 1]$ according to the Dixit-Stiglitz formula:

$$Y_t = \left[\int_0^1 Y_t(i)^{\frac{\phi_p - 1}{\phi_p}} di \right]^{\frac{\phi_p}{\phi_p - 1}},$$

where ϕ_p is a substitution elasticity among goods varieties. From the first-order condition of the final goods producers one obtains the demand for each of the intermediate goods, given the global demand:

$$Y_t(i) = \left(\frac{P_t(i)}{P_t} \right)^{-\phi_p} Y_t. \tag{3.3}$$

[3]Since the utility function is separable in all three arguments and there exist complete markets, the parts of the expected interteporal utility function, irrelevant for the wage and labour supply problem, can be omitted.

Zero-profit condition of the aggregators yields the final good price index:

$$P_t = \left[\int_0^1 P_t(i)^{1-\phi_p} di \right]^{\frac{1}{1-\phi_p}}.$$

The expression $\frac{\phi_p}{\phi_p - 1}$ can be interpreted as the markup of the intermediate goods producers.

3.1.4 Intermediate Good Producers

There exist a continuum of intermediate goods firms producing differentiated goods, $Y_t(i)$ using Cobb-Douglas technology:

$$Y_t(i) = \max\{Z_t^{1-\alpha} L_t(i)^{1-\alpha} K_t(i)^\alpha - Z_t \Phi, 0\},$$

where α is output elasticity with respect to capital[4]. The fixed cost Φ is included to ensure zero profits in the long run, in order to rule out entry into and exit from the intermediate goods market. Z_t is a nonstationary, labour-augmenting technology shock, common across all firms. Following Adolfson et al. (2007), technology growth rate is defined as $z_t^* = \frac{Z_t}{Z_{t-1}}$ and evolves according to an AR(1) process with mean γ^*:

$$z_t^* = (1 - \rho_z)\gamma^* + \rho_z z_{t-1}^* + \sigma_z \mu_{z,t}, \quad \text{where} \quad \mu_{z,t} \sim \mathcal{N}(0, 1).$$

All intermediate firms face the same input prices, therefore cost minimisation implies the same for all firms capital-labour ratio, proportional to the ratio of the factor prices:

$$\frac{K_t}{L_t} = \frac{\alpha}{1-\alpha} \frac{W_t}{R_t^k}.$$

Thus, the marginal cost MC_t is given by:

$$MC_t = \left(\frac{W_t}{(1-\alpha) Z_t} \right)^{1-\alpha} \left(\frac{R_t^k}{\alpha} \right)^\alpha.$$

Price setting

Because intermediate goods firms produce differentiated products, they are able to set their prices in a monopolistic fashion, i.e. they have possibility to add a mark-up over marginal cost[5]. The price setting is assumed to follow Calvo mechanism, i.e. there exist price rigidities. Each period only a fraction θ_p of intermediate firms is allowed to reoptimise their prices, while the remaining firms simply index their prices to the past inflation:

$$P_t = \pi_{t-1}^{\kappa_p} P_{t-1},$$

where κ_p stands for a degree of price indexation to past inflation. A firm which obtains a signal to reoptimise its price does it in a forward-looking manner, taking into consideration it might not be allowed to change its price for some time. Therefore, such a firm maximises its expected present discounted value

[4]It can be also interpreted as a capital's share of output.

[5]As in the classical producer problem from microeconomics, the mark-up is determined by the demand conditions, e.g. demand price elasticity.

of future profits over new price levels $\tilde{P}_t(i)$:

$$\max_{\tilde{P}_t(i)} \mathbb{E}_t \sum_{s=0}^{\infty} \theta_p^s \Xi_{t+s} \left(P_{t+s}(i) - MC_t\right) Y_{t+s}(i)$$

subject to:

$$\forall s \quad \text{eq.(3.3)},$$

$$P_{t+s}(i) = \begin{cases} \tilde{P}_t(i) & \text{for} \quad s = 0, \\ \tilde{P}_t(i) \prod_{k=1}^{s} (\pi_{t+k-1})^{\kappa_p} = \tilde{P}_t(i) \left(\frac{P_{t+s-1}}{P_{t-1}}\right)^{\kappa_p} & \text{for} \quad s > 0. \end{cases}$$

where $\Xi_{t+s} = \beta^s \Lambda_{t+s}$ is a discounting factor consisting of households' discounting rate and marginal utility of households' nominal income (exogenous to firms). The lack of firm-specific shocks in the model and equal expected marginal costs across the firms imply that all firms choose the same reoptimised price, which once again leads to a symmetric equilibrium. Therefore, the formula for the aggregate price level is given by:

$$P_t = \left[(1-\theta_p)\tilde{P}_t^{1-\phi_p} + \theta_p(\pi_{t-1}^{\kappa_p} P_{t-1})^{1-\phi_p}\right]^{\frac{1}{1-\phi_p}}.$$

3.1.5 Monetary Authorities

The central bank sets the short-run nominal interest rate R_t according to the Taylor rule:

$$\frac{R_t}{R} = \left(\frac{R_{t-1}}{R}\right)^{\kappa_r} \left[\left(\frac{\pi_t}{\pi}\right)^{\phi_\pi} \left(\frac{Y_t}{Y_t^*}\right)^{\phi_y}\right]^{1-\kappa_r} \exp\left(\varepsilon_{r,t}\right),$$

where R is the steady-state nominal interest rate, π is the stead-state inflation and Y_t^* is the target level of output. The latter is assumed to be equal to the trend level of output $Y_t^* = yZ_t$, where y is a the steady state output in terms of detrended variables. $\kappa_r \in [0,1]$ is a degree of interest rate smoothing; ϕ_y and ϕ_π are weights assigned by the central bank to deviations from steady-state values of output and inflation, respectively. $\varepsilon_{r,t}$ is a monetary policy shock, given by the exogenous process:

$$\varepsilon_{r,t} = \sigma_r \mu_{r,t} \quad \text{where} \quad \mu_{r,t} \sim \mathcal{N}(0,1).$$

3.1.6 Market clearing conditions:

The aggregate resource constraint is given by:

$$Y_t = C_t + I_t + G_t,$$

where:

$$C_t = \int_0^1 C_t(j)dj,$$
$$I_t = \int_0^1 I_t(j)dj,$$
$$K_t = \int_0^1 K_t(j)dj,$$
$$L_t = \left[\int_0^1 L_t(j)^{\frac{\phi_w-1}{\phi_w}} dj\right]^{\frac{\phi_w}{\phi_w-1}} = \int_0^1 L_t(i)di.$$

Since the model economy is a closed one, it is assumed that there is no fiscal policy, so that the government expenditure and trade balance are aggregated into an exogenous expenditure shock, to relate aggregate demand to aggregate supply of goods. In other words, the government spending shock is defined residually from the national income identity as a deviation from the stationary steady state government spending level:

$$\hat{g}_t = \varepsilon_{g,t},$$

with

$$\log \varepsilon_{g,t} = \rho_g \log \varepsilon_{g,t-1} + \sigma_g \mu_{g,t}, \quad \text{where} \quad \mu_{g,t} \sim \mathcal{N}(0,1).$$

3.2 Equilibrium Conditions

3.2.1 Households Sector

The Lagrangian for the household's optimisation problem is given by:

$$\begin{aligned}\mathcal{L}_{c,t}(j) = \mathbb{E}_t \sum_{s=0}^{\infty} \beta^s \Bigg\{ &\varepsilon_{d,t+s}\left[\log(C_{t+s}(j) - hC_{t+s-1}(j)) - \frac{\varepsilon_{l,t+s}}{1+\varphi} L_{t+s}(j)^{1+\varphi} + \varepsilon_{m,t+s} \log\left(\frac{M_{t+s}(j)}{Z_{t+s}P_{t+s}}\right)\right] \\ &+ \Lambda_{c,t+s}(j)\left[R_{t+s-1}B_{t+s-1}(j) + M_{t+s-1}(j) + \Pi_{t+s}(j) + W_{t+s}(j)L_{t+s}(j) + R^k_{t+s}K_{t+s}(j)\right. \\ &\left. - P_{t+s}C_{t+s}(j) - P_{t+s}I_{t+s}(j) - M_{t+s}(j) - B_{t+s}(j)\right)] \\ &+ \Lambda_{k,t+s}(j)\left[\varepsilon_{i,t+s}\left(1 - S\left(\frac{I_{t+s}(j)}{I_{t+s-1}(j)}\right)\right) I_{t+s}(j) + (1-\delta)K_{t+s}(j) - K_{t+s+1}(j)\right]\Bigg\},\end{aligned}$$

where $\Lambda_{c,t}(j)$ is the Lagrange multiplier on household's budget constraint[6] and $\Lambda_{k,t}(j)$ is the Lagrange multiplier on capital accumulation. The first-order conditions with respect to $C_t(j)$, $B_t(j)$, $K_{t+1}(j)$ and

[6]It can be interpreted as a marginal utility of consumption.

$I_t(j)$ are given by[7]:

$$\frac{\varepsilon_{d,t}}{C_t(j) - hC_{t-1}(j)} - \Lambda_{c,t}(j)P_t = h\beta\mathbb{E}_t\left\{\frac{\varepsilon_{d,t+1}}{C_{t+1}(j) - hC_t(j)}\right\} \tag{3.4}$$

$$R_t = \frac{1}{\beta}\mathbb{E}_t\left\{\frac{\Lambda_{c,t}(j)}{\Lambda_{c,t+1}(j)}\right\} \tag{3.5}$$

$$\Lambda_{k,t}(j) = \beta\mathbb{E}_t\left\{\Lambda_{k,t+1}(j)(1-\delta) + \Lambda_{c,t+1}(j)R^k_{t+1}\right\} \tag{3.6}$$

$$\beta\mathbb{E}_t\left\{\Lambda_{k,t+1}(j)\varepsilon_{i,t+1}\left(\frac{I_{t+1}(j)}{I_t(j)}\right)^2 S'\left(\frac{I_{t+1}(j)}{I_t(j)}\right)\right\} = -\Lambda_{k,t}(j)\varepsilon_{i,t}\left[1 - S\left(\frac{I_t(j)}{I_{t-1}(j)}\right) - \frac{I_t(j)}{I_{t-1}(j)}S'\left(\frac{I_t(j)}{I_{t-1}(j)}\right)\right] + \Lambda_{c,t}(j)P_t \tag{3.7}$$

The first two conditions, (3.4) and (3.5), imply stochastic Euler equation:

$$R_t = \frac{1}{\beta}\mathbb{E}_t\left\{\pi_{t+1}\frac{U_{c,t}(j) - \beta hU_{c,t+1}(j)}{U_{c,t+1}(j) - \beta hU_{c,t+2}(j)}\right\},$$

where $U_{c,t}(j)$ is a marginal utility of consumption:

$$U_{c,t}(j) = \frac{\partial U_t(j)}{\partial C_t(j)} = \frac{\varepsilon_{d,t}}{C_t(j) - hC_{t-1}(j)}.$$

The last two equations, (3.6) and (3.7), can be rewritten in terms of the price of installed capital which is defined as $Q_t(j) = \frac{\Lambda_{k,t}(j)}{\Lambda_{c,t}(j)P_t}$:

$$Q_t(j) = \mathbb{E}_t\left\{(1-\delta)\frac{Q_{t+1}(j)}{R_t}\frac{P_{t+1}}{P_t} + \frac{R^k_{t+1}}{R_tP_t}\right\},$$

$$1 - \varepsilon_{i,t}Q_t\left[1 - S\left(\frac{I_t(j)}{I_{t-1}(j)}\right) - \frac{I_t(j)}{I_{t-1}(j)}S'\left(\frac{I_t(j)}{I_{t-1}(j)}\right)\right] = \mathbb{E}_t\left\{\varepsilon_{i,t+1}\frac{Q_{t+1}}{R_t}\frac{P_{t+1}}{P_t}\left(\frac{I_{t+1}(j)}{I_t(j)}\right)^2 S'\left(\frac{I_{t+1}(j)}{I_t(j)}\right)\right\}.$$

3.2.2 Labour Market

The Lagrangian for the household's wage setting problem is given by:

$$\mathcal{L}_{w,t}(j) = \mathbb{E}_t\sum_{s=0}^{\infty}(\theta_w\beta)^s\left\{\Lambda_{c,t+s}(j)W_{t+s}L_{t+s}\left(\frac{\tilde{W}_t(j)}{W_{t+s}}\left(\frac{P_{t+s-1}}{P_{t-1}}\right)^{\kappa_w}\right)^{1-\phi_w} - \frac{\varepsilon_{d,t+s}\varepsilon_{l,t+s}}{1+\varphi}\left[L_{t+s}\left(\frac{\tilde{W}_t(j)}{W_{t+s}}\left(\frac{P_{t+s-1}}{P_{t-1}}\right)^{\kappa_w}\right)^{-\phi_w}\right]^{1+\varphi}\right\}.$$

The first order condition with respect to $\tilde{W}_t(j)$ is:

$$\mathbb{E}_t\sum_{s=0}^{\infty}(\theta_w\beta)^s L_{t+s}(j)\left\{\frac{\tilde{W}_t(j)}{P_{t+s}}\left(\frac{P_{t+s-1}}{P_{t-1}}\right)^{\kappa_w}(U_{c,t+s}(j) - \beta hU_{c,t+s+1}(j)) - \frac{\phi_w}{\phi_w - 1}U_{l,t+s}(j)\right\} = 0,$$

[7] Maximisation with respect to money balances comes from the budget constraint - the real money demand is characterised by optimisation decisions with respect to the remaining variables.

where $U_{l,t}(j)$ is a marginal disutility of labour:

$$U_{l,t}(j) = \frac{\partial U_t(j)}{\partial L_t(j)} = -\varepsilon_{d,t}\varepsilon_{l,t}L_t(j)^{\varphi}.$$

3.2.3 Intermediate Goods Producers

The Lagrangian for the firm's price setting problem is given by:

$$\mathcal{L}_{p,t}(i) = \mathbb{E}_t \sum_{s=0}^{\infty} \theta_p^s \Xi_{t+s} \left[\tilde{P}_t(i) \left(\frac{P_{t+s-1}}{P_{t-1}} \right)^{\kappa_p} - MC_{t+s} \right] \left[\frac{\tilde{P}_t(i)}{P_{t+s}} \left(\frac{P_{t+s-1}}{P_{t-1}} \right)^{\kappa_p} \right]^{-\phi_p} Y_{t+s},$$

and the related first order condition with respect to $\tilde{P}_t(j)$ is:

$$\mathbb{E}_t \sum_{s=0}^{\infty} \theta_p^s \Xi_{t+s} Y_{t+s} \left[\tilde{P}_t(i) \left(\frac{P_{t+s-1}}{P_{t-1}} \right)^{\kappa_p} - \frac{\phi_p}{\phi_p - 1} MC_{t+s} \right] \left[\frac{\tilde{P}_t(i)}{P_{t+s}} \left(\frac{P_{t+s-1}}{P_{t-1}} \right)^{\kappa_p} \right]^{-\phi_p} = 0,$$

which can be rewritten as:

$$\mathbb{E}_t \sum_{s=0}^{\infty} \theta_p^s \Xi_{t+s} Y_{t+s}(i) \left[\tilde{P}_t(i) \left(\frac{P_{t+s-1}}{P_{t-1}} \right)^{\kappa_p} - \frac{\phi_p}{\phi_p - 1} MC_{t+s} \right] = 0.$$

3.2.4 The Equilibrium

A definition of the equilibrium of the economy in question is standard and the equilibrium policy functions are determined by the following equations:

a) The household's first order conditions:

$$\Lambda_{c,t}(j)P_t = \frac{\varepsilon_{d,t}}{C_t(j) - hC_{t-1}(j)} - h\beta\mathbb{E}_t \left\{ \frac{\varepsilon_{d,t+1}}{C_{t+1}(j) - hC_t(j)} \right\}$$

$$R_t = \frac{1}{\beta}\mathbb{E}_t \left\{ \frac{\Lambda_{c,t}(j)}{\Lambda_{c,t+1}(j)} \right\}$$

$$Q_t(j) = \mathbb{E}_t \left\{ (1-\delta)\frac{Q_{t+1}(j)}{R_t}\frac{P_{t+1}}{P_t} + \frac{R_{t+1}^k}{R_t P_t} \right\},$$

$$1 - \varepsilon_{i,t}Q_t(j) \left[1 - S\left(\frac{I_t(j)}{I_{t-1}(j)} \right) - \frac{I_t(j)}{I_{t-1}(j)} S'\left(\frac{I_t(j)}{I_{t-1}(j)} \right) \right]$$
$$= \mathbb{E}_t \left\{ \varepsilon_{i,t+1}\frac{Q_{t+1}(j)}{R_t}\frac{P_{t+1}}{P_t} \left(\frac{I_{t+1}(j)}{I_t(j)} \right)^2 S'\left(\frac{I_{t+1}(j)}{I_t(j)} \right) \right\}$$

$$\mathbb{E}_t \sum_{s=0}^{\infty} (\theta_w\beta)^s L_{t+s}(j) \left\{ \frac{\tilde{W}_t(j)}{P_{t+s}} \left(\frac{P_{t+s-1}}{P_{t-1}} \right)^{\kappa_w} (U_{c,t+s}(j) - \beta h U_{c,t+s+1}(j)) - \frac{\phi_w}{\phi_w - 1} U_{l,t+s}(j) \right\} = 0,$$

b) The intermediate firm's first order conditions:

$$\frac{K_t(i)}{L_t(i)} = \frac{\alpha}{1-\alpha}\frac{W_t}{R_t^k},$$
$$MC_t = \left(\frac{W_t}{(1-\alpha)Z_t}\right)^{1-\alpha}\left(\frac{R_t^k}{\alpha}\right)^{\alpha},$$
$$\mathbb{E}_t \sum_{s=0}^{\infty} \theta_p^s \Xi_{t+s} Y_{t+s}(i)\left[\tilde{P}_t(i)\left(\frac{P_{t+s-1}}{P_{t-1}}\right)^{\kappa_p} - \frac{\phi_p}{\phi_p - 1} MC_{t+s}\right] = 0,$$

c) Technological constraints:

$$K_{t+1}(j) = (1-\delta)K_t(j) + \varepsilon_{i,t}\left(1 - S\left(\frac{I_t(j)}{I_{t-1}(j)}\right)\right) I_t(j),$$
$$Y_t(i) = Z_t^{1-\alpha} L_t(i)^{1-\alpha} K_t(i)^{\alpha} - Z_t \Phi,$$

d) The wage and price indices evolution equations:

$$W_t = \left[(1-\theta_w)\tilde{W}_t(j)^{1-\phi_w} + \theta_w(\pi_{t-1}^{\kappa_w} W_{t-1})^{1-\phi_w}\right]^{\frac{1}{1-\phi_w}},$$
$$P_t = \left[(1-\theta_p)\tilde{P}_t(j)^{1-\phi_p} + \theta_p(\pi_{t-1}^{\kappa_p} P_{t-1})^{1-\phi_p}\right]^{\frac{1}{1-\phi_p}},$$

e) The Taylor rule of the monetary authorities:

$$\frac{R_t}{R} = \left(\frac{R_{t-1}}{R}\right)^{\kappa_r}\left[\left(\frac{\pi_t}{\pi}\right)^{\phi_\pi}\left(\frac{Y_t}{Y_t^*}\right)^{\phi_y}\right]^{1-\kappa_r} \exp(\varepsilon_{r,t}),$$

f) Market clearing conditions:

$$Y_t = C_t + I_t + G_t,$$
$$C_t = \int_0^1 C_t(j)dj,$$
$$I_t = \int_0^1 I_t(j)dj,$$
$$K_t = \int_0^1 K_t(j)dj = \int_0^1 K_t(i)di$$
$$L_t = \left[\int_0^1 L_t(j)^{\frac{\phi_w - 1}{\phi_w}} dj\right]^{\frac{\phi_w}{\phi_w - 1}} = \int_0^1 L_t(i)di.$$

Owing to the symmetry of the equilibrium, in further the indexes j and i will be omitted.

3.2.5 Stochastic processes

There are several sources of disturbances in the model economy, all of which are assumed to be independent of one another. The first and the most important concerns the labour-augmenting technology Z_t and introduces non-stationarity into the model. Since Z_t follows a unit root process, the growth rate of the technology z_t^* follows an AR(1) stationary process, and the mean value of z_t^* is γ^*. The deviation of z_t^* from this steady state value, denoted $\hat{z}_t^*$, is given by:

$$\hat{z}_t^* = \rho_z \hat{z}_{t-1}^* + \frac{\sigma_z}{\gamma^*} \mu_{z,t}$$

Preference shifts $\varepsilon_{d,t}$, $\varepsilon_{l,t}$, $\varepsilon_{m,t}$, investment cost disturbance $\varepsilon_{i,t}$ and government spending shock $\varepsilon_{g,t}$ follow AR(1) stationary processes in logs:

$$\log \varepsilon_{\iota,t} = \rho_\iota \log \varepsilon_{\iota,t-1} + \sigma_{\iota i} \mu_{\iota,t}.$$

However, due to the absence of equilibrium condition for the money holdings, the money demand shock can be neglected. The unconditional mean of $\varepsilon_{\iota,t}$ is equal to 1 ($\iota \in \{d, l, i, g\}$). The log deviation of $\varepsilon_{\iota,t}$ from its steady-state, denoted $\hat{\varepsilon}_{\iota,t}$, is thus given by:

$$\hat{\varepsilon}_{\iota,t} = \rho_\iota \hat{\varepsilon}_{\iota,t-1} + \sigma_{\iota i} \mu_{\iota,t}.$$

The monetary policy shock $\varepsilon_{r,t}$ is a white noise process and therefore its expected value is equal to 0.

3.3 Model Solution

The first order conditions, together with the equilibrium conditions described in the previous section, form a system of nonlinear rational expectations system, which has no solution that can be derived analytically, and thus needs to be found with numerical methods. As stated in An and Schorfheide (2007), in the context of likelihood-based DSGE model estimation, linear approximation techniques are very popular because they lead to a state-space representation of the DSGE model that can be analysed with the Kalman filter. Therefore, the model will be log-linearised around the deterministic steady state, which will enable to solve the model, leading to its state-space representation.

3.3.1 Stationary Equilibrium

Since the model has a unit root in the technology process Z_t, the analysed economy evolves along stochastic growth path. Therefore, as in the work of Adolfson et al. (2007), non-stationary variables will be detrended as follows:

$$y_t = \frac{Y_t}{Z_t}, \qquad c_t = \frac{C_t}{Z_t}, \qquad i_t = \frac{I_t}{Z_t}$$
$$k_{t+1} = \frac{K_{t+1}}{Z_t}, \qquad g_t = \frac{G_t}{Z_t}, \qquad w_t = \frac{W_t}{P_t Z_t}.$$

R_t, π_t and L_t remain unchanged as they are stationary. Moreover, for convenience, some variables will be rescaled:

$$r_t^k = \frac{R_t^k}{P_t}, \qquad mc_t = \frac{MC_t}{P_t},$$
$$\lambda_{c,t} = \Lambda_{c,t} Z_t P_t, \qquad \lambda_{k,t} = \Lambda_{k,t} Z_t P_t.$$

3.3.2 Steady states

The model economy has a unique steady state in terms of the detrended variables, which is achieved when stochastic disturbances are continuously equal to zero. For simplicity, the following assumptions concerning steady-states values of inflation and fixed cost have been made: $\pi = 1$ and $\Phi = 0$. The steady-states of the remaining model variables have been derived from the stationarised model equations and are given by:

$$\lambda_c = \frac{z^* - \beta h}{c(z^* - h)}, \qquad R = \frac{z^*}{\beta},$$
$$Q = \frac{r^k}{R - (1-\delta)}, \qquad Q = 1,$$
$$\lambda_c \tilde{w} = \frac{\phi_w}{\phi_w - 1} U_l, \qquad U_l = -L^{\varphi},$$
$$\frac{k}{z^* L} = \frac{\alpha}{1-\alpha} \frac{w}{r^k}, \qquad w = \tilde{w},$$
$$mc = \left(\frac{w}{1-\alpha}\right)^{1-\alpha} \left(\frac{r^k}{\alpha}\right)^{\alpha}, \qquad mc = \frac{\phi_p - 1}{\phi_p},$$
$$i = \Big(z^* - (1-\delta)\Big) k, \qquad P = \tilde{P}$$
$$y = c + i + g, \qquad y = L^{1-\alpha} \left(\frac{k}{z^*}\right)^{\alpha}.$$

3.3.3 Log-linearised model equations

In further, the following notation convention is used:

$$\hat{x}_t = \log x_t - \log x,$$

i.e. $\hat{x}_t$ stands for the log deviation of x_t from its steady state value, x. After log-linearisation, the formulas from the Section 3.2.4 are given as follows[8]:

[8] A more detailed derivation of each equation is available in the appendix B.

a) The household's first order conditions:

$$(z^* - \beta h)(z^* - h)\hat{\lambda}_{c,t} = z^* h\hat{c}_{t-1} - \left((z^*)^2 + \beta h^2\right)\hat{c}_t + \beta z^* h\mathbb{E}_t\{\hat{c}_{t+1}\}$$
$$- z^* h\hat{z}^*_t + \beta z^* h\mathbb{E}_t\{\hat{z}^*_{t+1}\} + z^*(z^* - h)\hat{\varepsilon}_{d,t} - \beta h(z^* - h)\mathbb{E}_t\{\hat{\varepsilon}_{d,t+1}\}, \tag{3.8}$$

$$\hat{R}_t = \hat{\lambda}_{c,t} - \mathbb{E}_t\{\hat{\lambda}_{c,t+1}\} + \mathbb{E}_t\{\hat{\pi}_{t+1}\} + \mathbb{E}_t\{\hat{z}^*_{t+1}\}, \tag{3.9}$$

$$\hat{Q}_t = -\hat{R}_t + \mathbb{E}_t\{\hat{\pi}_{t+1}\} + \frac{1-\delta}{R}\mathbb{E}_t\{\hat{Q}_{t+1}\} + \left(1 - \frac{1-\delta}{R}\right)\mathbb{E}_t\{\hat{r}^k_{t+1}\}, \tag{3.10}$$

$$\hat{i}_t = \frac{1}{1+\beta}\hat{i}_{t-1} + \frac{\beta}{1+\beta}\mathbb{E}_t\{\hat{i}_{t+1}\} - \frac{1}{1+\beta}\hat{z}^*_t - \frac{\beta}{1+\beta}\mathbb{E}_t\{\hat{z}^*_{t+1}\} + \frac{\hat{Q}_t + \hat{\varepsilon}_{i,t}}{(1+\beta)(z^*)^2 S''}, \tag{3.11}$$

$$\mathbb{E}_t\sum_{s=0}^{\infty}(\theta_w\beta)^s\left\{\hat{\lambda}_{c,t+s} + \hat{\tilde{w}}_t + \kappa_w(\hat{P}_{t+s-1} - \hat{P}_{t-1}) - \hat{U}_{l,t+s}\right\} = 0 \tag{3.12}$$

$$\text{where: } U_l = -L^{\varphi}, \quad \hat{U}_{l,t} = \hat{\varepsilon}_{d,t} + \hat{\varepsilon}_{l,t} + \varphi\hat{L}_t,$$

b) The intermediate firm's first order conditions:

$$\hat{r}^k_t = \hat{L}_t + \hat{w}_t + \hat{z}^*_t - \hat{k}_t, \tag{3.13}$$

$$\hat{mc}_t = (1-\alpha)\hat{w}_t + \alpha\hat{r}^k_t, \tag{3.14}$$

$$\mathbb{E}_t\sum_{s=0}^{\infty}(\theta_p\beta)^s\left(\hat{\tilde{P}}_t - \hat{P}_{t+s} + \kappa_p(\hat{P}_{t+s-1} - \hat{P}_{t-1}) - \hat{mc}_{t+s}\right) = 0, \tag{3.15}$$

c) Technological constraints:

$$\hat{k}_{t+1} = \frac{1-\delta}{z^*}\left(\hat{k}_t - \hat{z}^*_t\right) + \left(1 - \frac{1-\delta}{z^*}\right)\left(\hat{i}_t + \hat{\varepsilon}_{i,t}\right), \tag{3.16}$$

$$\hat{y}_t = (1-\alpha)\hat{L}_t + \alpha\left(\hat{k}_t - \hat{z}^*_t\right), \tag{3.17}$$

d) The wage and price indices evolution equations:

$$\hat{w}_t = (1-\theta_w)\hat{\tilde{w}}_t + \theta_w(\hat{w}_{t-1} + \kappa_w\hat{\pi}_{t-1} - \hat{\pi}_t - \hat{z}^*_t), \tag{3.18}$$

$$\hat{P}_t = (1-\theta_p)\hat{\tilde{P}}_t + \theta_p(\hat{P}_{t-1} + \kappa_p\hat{\pi}_{t-1}), \tag{3.19}$$

e) The Taylor rule of the monetary authorities:

$$\hat{R}_t = \kappa_r\hat{R}_{t-1} + (1-\kappa_r)(\phi_\pi\hat{\pi}_t + \phi_y\hat{y}_t) + \varepsilon_{r,t}, \tag{3.20}$$

f) Market clearing condition:

$$\hat{y}_t = \frac{c}{y}\hat{c}_t + \frac{i}{y}\hat{i}_t + \frac{g}{y}\hat{\varepsilon}_{g,t}. \tag{3.21}$$

3.3.4 Derived pricing equations

From the above formulas one can obtain pricing equations with a standard economic interpretation. These are:

- Phillips curve:
  The standard general new-Keynesian Phillips curve (NKPC) is obtained by combining (3.15) for t and $t+1$ with (3.14) and (3.19):

$$\hat{\pi}_t = \frac{\beta}{1+\beta\kappa_p}\mathbb{E}_t\{\hat{\pi}_{t+1}\} + \frac{\kappa_p}{1+\beta\kappa_p}\hat{\pi}_{t-1} + \frac{1-\theta_p}{\theta_p}\frac{1-\beta\theta_p}{1+\beta\kappa_p}\Big((1-\alpha)\hat{w}_t + \alpha\hat{r}_t^k\Big). \tag{3.22}$$

  Therefore, present inflation depends on expected future inflation, past inflation as well as on the present marginal cost. The partial indexation parameter κ_p determines degree of backward-lookingness of inflation, whereas the price flexibility parameter θ_p - degree of price stickness.

- Wage equation:
  Similarly to the NKPC, one can derive the real wage equation by combining (3.12) for t and $t+1$ with (3.18):

$$\begin{aligned}\hat{w}_t = \ & \frac{\theta_w}{1+\beta(\theta_w)^2}\Big[\beta\mathbb{E}_t\{\hat{w}_{t+1}\} + \hat{w}_{t-1} - (1+\beta\kappa_w)\hat{\pi}_t - \beta\theta_w\mathbb{E}_t\{\hat{\pi}_{t+1}\} + \kappa_w\hat{\pi}_{t-1} \\ & -\hat{z}_t^* + \beta\theta_w\mathbb{E}_t\{\hat{z}_{t+1}^*\} + \frac{1-\theta_w}{\theta_w}(1-\beta\theta_w)\Big(\hat{U}_{l,t} - \hat{\lambda}_{c,t}\Big)\Big]\end{aligned} \tag{3.23}$$

  The NKPC alike, the dependance of the real wage on its own past value is characterised by the stickness parameter θ_w, and on the past inflation - by the partial indexation parameter κ_w.

3.3.5 Model forms

Structural and reduced form

The above 11 log-linearised model equations, (3.8)-(3.11), (3.13), (3.20)-(3.23), form a system of linear stochastic difference equations called the "structural form" of the DSGE model, which can be expressed as follows:

$$\begin{cases} \mathbb{E}_t\Big\{\tilde{\Gamma}_{-1}s_{t+1} + \tilde{\Gamma}_0 s_t + \tilde{\Gamma}_1 s_{t-1} + \tilde{\Gamma}_{\varepsilon,-1}\varepsilon_{t+1} + \tilde{\Gamma}_\varepsilon\varepsilon_t\Big\} = 0, \\ \varepsilon_t = \rho\varepsilon_{t-1} + \sigma\mu_t, \end{cases} \tag{3.24}$$

where s_t stands for the vector of endogenous variables (state variables):

$$s_t = [\hat{y}_t, \hat{c}_t, \hat{i}_t, \hat{k}_t, \hat{R}_t, \hat{r}^k, \hat{L}_t, \hat{w}_t, \hat{\pi}_t, \hat{Q}_t, \hat{\lambda}_t]',$$

ε_t denotes the vector of exogenous shocks:

$$\varepsilon_t = [\hat{z}_t^*, \hat{\varepsilon}_{d,t}, \hat{\varepsilon}_{l,t}, \hat{\varepsilon}_{i,t}, \hat{\varepsilon}_{g,t}, \hat{\varepsilon}_{r,t}]',$$

the matrices $\tilde{\Gamma}_{-1}$, $\tilde{\Gamma}_0$, $\tilde{\Gamma}_1$, $\tilde{\Gamma}_{\varepsilon,-1}$ and $\tilde{\Gamma}_\varepsilon$ stack parameters of the log-linearised equations and the vectors ρ and σ consist of the parameters of the stochastic processes. The system (3.24) can be expressed equivalently as:

$$\Gamma_0 s_t = \Gamma_1 s_{t-1} + \Gamma_\varepsilon \varepsilon_t + \Gamma_\eta \eta_t, \tag{3.25}$$

where η_t is a vector of rational expectations forecast errors $\eta_{x,t}$, defined as $\eta_x = \hat{x}_t - \mathbb{E}_{t-1}\{\hat{x}_t\}$, where x is a forward-looking variable. Then to solve (3.25), means to rewrite it in a reduced form of:

$$s_t = \Phi_1 s_{t-1} + \Phi_\varepsilon \varepsilon_t, \tag{3.26}$$

which implies finding the matrices Φ_1 and Φ_ε. In practise, to deal with this issue, the numerical methods are being used, e.g. the Dynare package[9], which are based on the generalised Schur decomposition (QZ decomposition) of the matrices Γ_0 and Γ_1[10]. The equation (3.26) is also called the "state transition equation", because it describes how the system passes from the state in the moment t to a state in the moment $t+1$.

State-space form

The vector s_t gathers variables which are theoretical and rarely can be observed directly. Therefore, to estimate the original model, the relationship between conceptual and statistical variables has to be found. It is done within a linear state-space form, which is then used to evaluate the likelihood function using the Kalman Filter. The state-space form comprises of the transition equation (3.26) and the "measurement equation" given by:

$$y_t = A + Bs_t. \tag{3.27}$$

As stated in Lubik and Schorfheide (2006), the measurement equation links the model variables s_t to the vector of observed variables y_t[11] through the matrices A and B. The former consists of the mean values of y_t - related to the underlying structural parameters, while the latter selects the elements of s_t and thus does not depend on model parameters. Since the exact form of the measurement equation depends on the choice of observables, the details of the measurement equation will be presented in the empirical part of this paper.

[9] The code of the Dynare `dsge.mod` file, used to solve the model presented in this paper, can be found in the Appendix C.

[10] The details of the algorithm can be found e.g. in Sims (2002).

[11] For the sake of coherence with the DSGE-VAR part, the vector of observables is also denoted y_t, which should not be confused with the stationarised output.

Chapter 4

Empirical Analysis

4.1 Data

To estimate the model, six key macroeconomic quarterly observable variables for Poland are used[1]: real GDP (GDP_t), real consumption (final consumption expenditure of households, $CONS_t$), real investment (gross fixed capital formation, INV_t), real wage (compensation of employees, WG_t), inflation rate (chained CPI inflation rate, CPI_t) - all seasonally adjusted with X-12-ARIMA method - and the nominal short-term interest rate (3 month WIBOR, $WIBOR_t$). The period covered is 1995Q1 - 2010Q4, which gives 64 observations for each variable.

The time series of GDP, consumption, investment, wage and the interest rate were taken from the Eurostat online database. The series of population 16 years and older came from the OECD online database. The inflation rate series was taken from the GUS official web page. The inflation target series was constructed basing on the NBP official announcements.

The nominal series of the GDP, consumption and investment were turned into the real terms by deflating them with the CPI deflator[2] and dividing by population 16 years and older. The real wage was obtained by deflating the nominal wage using the CPI deflator. The inflation rate as well as the interest rate were corrected for the inflation target, priorly smoothened using centred moving average. Due to the model assumption of the presence of a deterministic growth rate γ (common for all real variables), the data has not been detrended prior to estimation.

After the transformations described above, the log differences of all real variables were taken to obtain their quarterly growth rates, which were thereafter converted into percentages. The interest rate has been expressed in quarterly terms. Figure D.1 in the Appendix D presents the plots of the transformed data.

[1]The symbols of the corresponding observables, together with captions of the statistical series when needed, are given in the brackets.

[2]Consumer price index with 1995Q1 as a base period.

The corresponding measurement equations is given by:

$$y_t \equiv \begin{bmatrix} Y_t^{obs} \\ C_t^{obs} \\ I_t^{obs} \\ W_t^{obs} \\ \Pi_t^{obs} \\ R_t^{obs} \end{bmatrix} = \begin{bmatrix} 100 \times \Delta \log GDP_t \\ 100 \times \Delta \log CONS_t \\ 100 \times \Delta \log INV_t \\ 100 \times \Delta \log WG_t \\ CPI_t \\ 0.25 \times WIBOR_t \end{bmatrix} = \begin{bmatrix} 100 \times (\hat{y}_t - \hat{y}_{t-1} + \hat{z}_t^* + \log \gamma^*) \\ 100 \times (\hat{c}_t - \hat{c}_{t-1} + \hat{z}_t^* + \log \gamma^*) \\ 100 \times (\hat{i}_t - \hat{i}_{t-1} + \hat{z}_t^* + \log \gamma^*) \\ 100 \times (\hat{w}_t - \hat{w}_{t-1} + \hat{z}_t^* + \log \gamma^*) \\ 100 \times (\hat{\pi}_t + \bar{\pi}) \\ 100 \times (\hat{R}_t + \bar{R}) \end{bmatrix},$$

where Δ denotes first time difference of a variable. Because the steady-state inflation rate is assumed to be equal to one[3], $\bar{\pi} = \log \pi$ is set at zero; $\bar{R}$ is the steady-state interest rate, determined by the model parameters: $\bar{R} = \log R$, where $R = \gamma^*/\beta$.

Note on the absence of the labour input as an observable

There is a need to comment on the absence of the data concerning the labour input, as the hours worked are often included as an observable in the measurement equation[4]. The reasons for such a decision were both of theoretical and practical nature.

Firstly, a problem of singularity of the covariance matrix generated by the DSGE model might arise, when the number of structural shocks is not equal to the number of endogenous variables to which the model is fitted. As stated in Lubik and Schorfheide (2006), any DSGE model that generates a rank-deficient covariance matrix for y_t is clearly at odds with the data, since in the forecast error covariance matrix of VAR models is non-singular (because y_t is predicted based on its lagged values). Hence, the larger the dimension of y_t, the more shocks have to be introduced into the model. Therefore, it was desirable to make the number of shocks and the number of observables equal, and since there are six exogenous shocks in the DSGE model, the number of observables should be set to six as well.

Obviously, the same goal could have been achieved by introducing an additional shock, as it is recommended in Del Negro and Schorfheide (2004). However, it has been considered as unsuitable due to the scarcity of the data available for Poland - both in terms of shortness of the time series and the lack of proper counterpart for the "hours worked" data. Moreover, the employment in the Polish economy, due to its specificity, deserves a separate treatment and adding labour input data into the model, without former in-depth analysis, could lead to misleading results and cast doubts over presented outcomes.

4.2 Parameters

The assumptions concerning model parameters, i.e. either their calibrated values, or the specifications for their prior distributions, are taken mainly from the studies of the Polish economy: Grabek et al. (2007)

[3]In the model, the central bank follows the zero inflation target. The fact, that the inflation rate together with the interest rate were detrended with the official inflation target of the NBP is compatible with this assumption.

[4]See Smets and Wouters (2003), Del Negro et al. (2007), Watanabe (2009), Adjemian et al. (2008).

and Kolasa (2009). In several cases slight corrections were needed to adjust the assumptions either to the long-term restrictions imposed by the data or to the measurement equations.

4.2.1 Calibrated parameters

Several model parameters characterise only the steady state, i.e. the long-term relationships, and as such are not identifiable from the data and cannot be estimated on log differences of the time series. Hence, it is a common practice in the DSGE modelling to calibrate several of them[5]. The fact that the available sample is rather short additionally encourages to resort to such a method of setting parameters. The exact calibrated values are reported in the table 4.1.

Table 4.1: Calibrated parameters

Symbol	Interpretation	Value
α	share of capital in output	0.330
β	household's discount factor	0.995
δ	depreciation rate	0.025
ϕ_w	substitution elasticity among labour varieties	3.000
ϕ_p	substitution elasticity among good varieties	3.000
	Steady state shares in output	
c/y	consumption-output ratio	0.582
g/y	government spending-output ratio	0.200

The calibrated values of the steady state shares in output of consumption and government spending imply the steady state share of investment equal to 0.218, which in turn leads to the steady state capital-output ratio of 8.720.[6]

4.2.2 Prior distributions

The remaining parameters, assumed to be independently distributed, are estimated with Bayesian techniques. The parameters from the $[0, 1]$ interval are assumed to be beta distributed. These are: habit persistence parameter h, Calvo probabilities θ_w and θ_p, indexation parameters κ_w and κ_p, interest smoothing parameter κ_r as well as, to ensure stationarity of stochastic processes, the shock persistence parameters, $\rho_\iota,\ \iota \in \{z, d, l, i, g\}$. The gamma distribution is used for the parameters taking only positive values: inverse labour supply elasticity φ, the Taylor rule parameters ϕ_π and ϕ_y and the technology growth rate γ. For the shock standard deviations $\sigma_\iota,\ \iota \in \{z, d, l, i, g, m\}$ the inverse gamma distribution has been used. The adjustment cost parameter S'' is assumed to follow a normal distribution.

Moreover, as already mentioned, following Adjemian et al. (2008) the DSGE model prior weight was estimated and the uniform distribution was used as a prior for λ, since there were no strong initial beliefs

[5]See Grabek et al. (2007), Kolasa (2009), Pytlarczyk (2005).
[6]This is a straightforward derivation from the steady state relations described in the previous chapter.

concerning the optimal size of the artificial sample. However, the distribution support was tighter than in the cited paper and was limited to the interval $[0, 2]$, because from the beginning on the model has been expected to be significantly misspecified.

The details concerning prior distributions are presented in the tables reporting on the estimation results. In most of the cases they are similar to those used in Grabek et al. (2007) and Kolasa (2009). In particular, the mean of the technology growth rate, assumed to be equivalent to the long-term economic growth rate, is set to 1.0085 implying a growth rate of 3.4% per annum - a value reported in Grabek et al. (2007).

4.3 Solution and estimation

The practical implementation, i.e. solving of the DSGE-VAR model and its Bayesian estimation, was carried out using the Dynare package, version 4.2.2 (Adjemian et al., 2011). The Dynare file code can be found in the Appendix C. The likelihood function was evaluated using the Kalman Filter. The considered lag length for the DSGE-VAR model was set equal to 4. In order to make the DSGE model estimation comparable with the VAR estimation, in the former case the first four observations have been excluded from the estimation. To obtain draws from the posterior distribution, 500,000 replications for the Metropolis-Hastings algorithm were run, with the burn-in period ratio of 0.5[7]. To find the mode of the posterior density, the `fmincon` optimisation routine[8] was used. The scaling parameter for the inverse Hessian computed at the obtained mode - used for the jumping distribution in the RWMH algorithm - was set equal to 0.45 yielding the average acceptation rate per chain of 0.26.

4.4 Results

The obtained estimation results are reported below - for the structural parameters in the table 4.4, and for the shock parameters in the table 4.3. The plots of the prior and posterior densities for all parameters can be found in the Appendix D.

[7] The ratio of the number of the first draws to be disregarded to the total number of runs.

[8] `fmincon` is a function included in MATLAB's Optimization Toolbox, used to minimise a scalar function of multiple variables in constrained optimisation problems with linear restrictions.

Table 4.2: Estimation results - structural parameters

Symbol	Interpretation	Prior distribution			Optimisation result		Posterior distribution		
		Type	Mean	St. dev.	Mode	St. dev.	Mean	5%	95%
h	habit persistence	beta	0.70	0.10	0.703	0.063	0.716	0.619	0.815
φ	inv. labour supply elast.	gamma	2.00	0.70	1.209	0.438	1.426	0.641	2.170
S''	capital adjustment cost	normal	6.00	1.50	4.336	1.218	5.085	2.927	7.243
θ_w	Calvo wages	beta	0.60	0.10	0.564	0.077	0.595	0.480	0.717
θ_p	Calvo prices	beta	0.60	0.10	0.618	0.044	0.619	0.547	0.692
κ_w	wage indexation	beta	0.50	0.15	0.542	0.173	0.533	0.290	0.780
κ_p	price indexation	beta	0.50	0.15	0.504	0.154	0.512	0.281	0.735
κ_r	interest rate smoothing	beta	0.80	0.08	0.835	0.026	0.835	0.791	0.877
ϕ_π	inflation response	gamma	1.70	0.15	1.829	0.145	1.846	1.613	2.089
ϕ_y	output response	gamma	0.125	0.065	0.043	0.026	0.059	0.013	0.104
γ	technology growth rate	gamma	1.0085	0.0015	1.0078	0.001	1.008	0.006	0.010
λ	DSGE prior weight	uniform	1.00	1.00	1.044	0.130	1.061	0.840	1.276

Table 4.3: Estimation results - shock parameters

Symbol	Prior distribution			Optimisation result		Posterior distribution		
	Type	Mean	St. dev.	Mode	St. dev.	Mean	5%	95%
Shock persistence parameters								
ρ_z	beta	0.40	0.10	0.171	0.056	0.188	0.092	0.276
ρ_d	beta	0.70	0.10	0.612	0.115	0.607	0.439	0.782
ρ_l	beta	0.70	0.10	0.439	0.101	0.435	0.279	0.593
ρ_i	beta	0.70	0.10	0.552	0.104	0.539	0.373	0.702
ρ_g	beta	0.70	0.10	0.768	0.096	0.741	0.595	0.892
Shock dispersion parameters								
σ_z	inv. gamma	0.005	inf	0.015	0.003	0.016	0.011	0.021
σ_d	inv. gamma	0.005	inf	0.027	0.005	0.030	0.020	0.048
σ_l	inv. gamma	0.005	inf	0.074	0.025	0.094	0.043	0.143
σ_i	inv. gamma	0.007	inf	0.080	0.026	0.100	0.050	0.147
σ_g	inv. gamma	0.010	inf	0.030	0.004	0.037	0.024	0.039
σ_r	inv. gamma	0.003	inf	0.0016	0.0002	0.0016	0.0012	0.0020

The posterior mode for λ hyperparameter is equal to 1.061, with the 90% highest density interval ranging from 0.840 to 1.276. It can be interpreted in the following way: the optimal mixed sample, used to estimate the VAR, consists of 63 actual observations and 67 artificial ones. Hence, the best outcome is achieved for the sample with more observations generated by the DSGE model than coming from the data. Moreover, the obtained value is remarkably higher than the minimum one (needed for the prior to be well defined), equal to 0.48. Therefore, one can infer that the DSGE model is a useful source of information for the VAR estimation. On the one hand, since any finite value of λ suggests that loosening of the restriction imposed by the structural model improves the empirical performance of the DSGE-VAR model, the estimated DSGE prior weight indicates significant misspecification of the presented DSGE model. Furthermore, the estimated value is quite low compared to the results of previously cited authors. For example, Adjemian et al. (2008) obtained values equal to 1.25 and 1.55 (for two model specifications) and Del Negro et al. (2007), who did not estimate λ but chose its value to maximise the marginal density, reported the value 1.25. However, in their earlier research, Del Negro et al. (2004) achieved considerably lower value of 0.75, the result reached also by Watanabe (2009).

Overall, the parameter estimates are in line with those obtained in previously cited studies of the Polish economy. Moreover, the comparison of the prior and posterior densities implies that the data are quite informative for the most of the parameters. However, in some cases both distributions almost coincide: for the indexation parameters and for the parameter of the government spending shock persistence the likelihood does not seem to contain any meaningful information.

The degree of external habit formation seems to be significant in Poland, with the posterior mode of h equal to 0.72, lying between the earlier results for the Polish economy (0.80 in Kolasa, 2009 and 0.60

in Grabek et al., 2007). The estimated mean of the inverse labour supply elasticity, equal to 1.43, is lower than the one reported in Kolasa (2009), 1.95, as well as in the studies of other economies (e.g. 1.69 in Del Negro et al., 2007 or 1.80 in Pytlarczyk, 2005). The found mean of the curvature of the capital adjustment cost function (5.08) is lower than the one obtained in earlier works on the Polish economy (e.g. 6.24 in Grabek et al., 2007), however still higher than, for instance, in Germany (2.42, Pytlarczyk, 2005) or in the US (4.57, Del Negro et al., 2007).

The posterior means of the Calvo probabilities (0.59 and 0.62 for wages and prices, respectively) are very similar to the values reported for the Polish economy by the previously cited authors and suggest that, on average, prices and wages are being reoptimised in Poland every two-and-half quarter. Therefore, the model implies moderate price and wage stickiness in Poland, in particular when compared to other economies. For example, the findings for Germany are 0.85 and 0.96 for wages and prices, respectively (Pytlarczyk, 2005), wheres Del Negro et al. (2007) report the posterior mean for the US economy of 0.79 for both. The posterior means for the wage and price indexation parameters are noticeably higher than in other studies on the Polish economy (0.53 and 0.51, respectively), where these are usually close to 0.30. It is of no wonder, however, taking into consideration their already mentioned poor identifiability from the data, which leads to the posterior coinciding with the prior.

According to the estimates, the monetary authorities in Poland follow a version of the Taylor rule, strongly reacting to inflation deviations from its target value and virtually neglecting output shifts. The posterior means are equal to 1.85 and 0.06 for the inflation and output response parameters, respectively. Moreover, interest rate smoothing seems to be of concern in the Polish monetary policy, with the relevant parameter of 0.83. These findings are quite similar to those obtained by other authors for the Polish economy: e.g. corresponding values reported by Grabek et al. (2007) are 1.38, 0.03 and 0.82.

As far as the disturbances are concerned, the attained estimates suggest moderate shock persistence and limited degree of disturbance volatility in the Polish economy and are roughly in line with the previously cited studies. The only exception is the technology process with the posterior mean of the autoregressive coefficient equal to 0.19, which is a considerably smaller value than e.g. 0.81 reported in Grabek et al. (2007). Posterior means of shock dispersion parameters are quite low when compared to the results for the Polish economy presented by the previously cited authors, yet still they exceed the estimates for other economies (e.g. Pytlarczyk, 2005). Such results may suggest that the severeness of shocks hitting the Polish economy has suppressed in recent years, assimilating it to more stable economies.

Chapter 5

Model evaluation

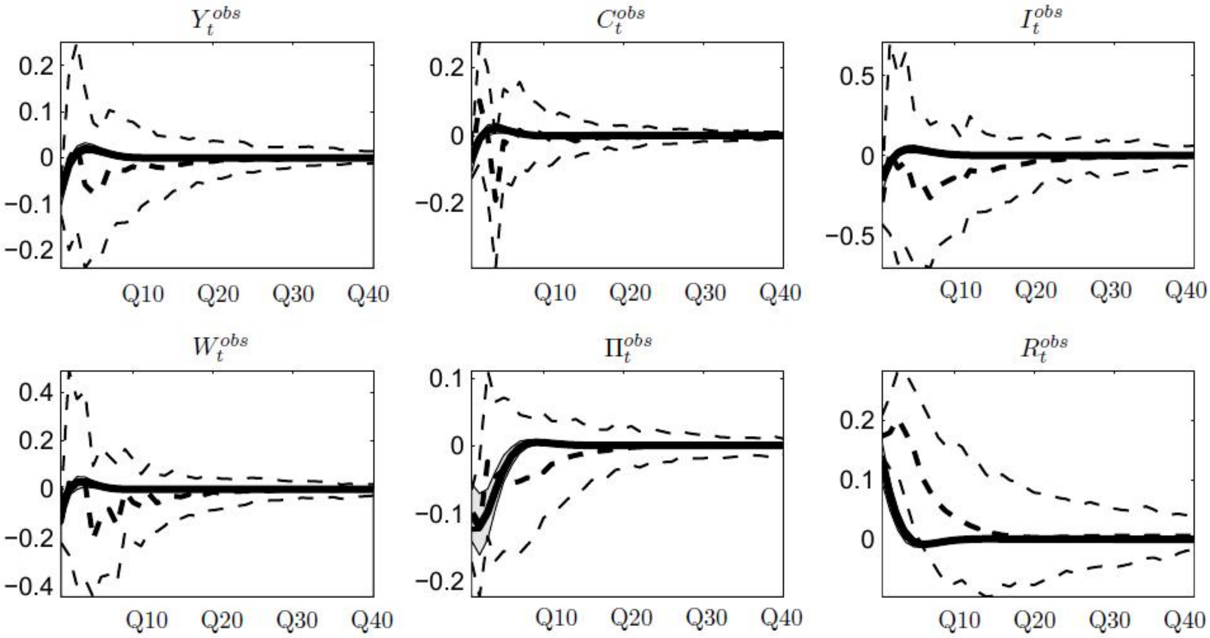
5.1 Impulse response functions

In order to assess the quality of the DSGE model, it is desirable to analyse what reactions of observed variables to stochastic disturbances it indicates. To do so, one can compare the impulse response functions (IRFs) obtained from the structural model with those delivered by the empirical one. However, reduced form VAR is not capable of "unpacking" the shocks which hit the system and therefore it needs to be identify. The identification procedure, described in the Section 2.4, enables computation of the IRFs in a Bayesian fashion, making use of the hybrid structure of the DSGE-VAR model. Such a procedure can be understood as comparing the prior IRFs, generated by the DSGE model, to the posterior IRFs, obtained after updating the initial beliefs with the information coming from the data. Following Del Negro et al. (2007), IRFs with respect to only two kinds of shocks will be presented in detail, i.e. to a monetary policy shock and to a technology one, since these are the most important disturbances for policy analysis. The remaining IRFs are presented in the Appendix D.

Figure 5.1 reveals the unsatisfactory performance of the DSGE model with respect to identification of responses to a monetary policy shock. The theoretical model suggests that growth rates of output, consumption, investment and wages react almost instantaneously to the monetary disturbance and that these reactions fade rapidly - contrary to the DSGE-VAR model, which indicates a hump-shape responses. The IRFs delivered by the latter show that the impact of monetary policy disturbance is sizable and rather long-lasting, especially in the case of investment and wages growth rates. The inflation and the interest rate responses implied by the DSGE model match those from the DSGE-VAR model, yet, again, their persistence is much smaller. As far as the uncertainty of the estimates is concerned, the reactions implied by the DSGE-VAR model are less precise than those from the DSGE model, in which case the 90% intervals almost coincide with the mean estimates. However, the IRFs from the DSGE model are virtually always within the 90% bands delivered by the DSGE-VAR model.

Figure 5.2 shows that, contrary to reactions to a monetary policy shock, the IRFs with respect to a technology disturbance almost coincide for both models. Thus, the constructed structural model matches

Figure 5.1: Bayesian Impulse Response Function - monetary policy shock.



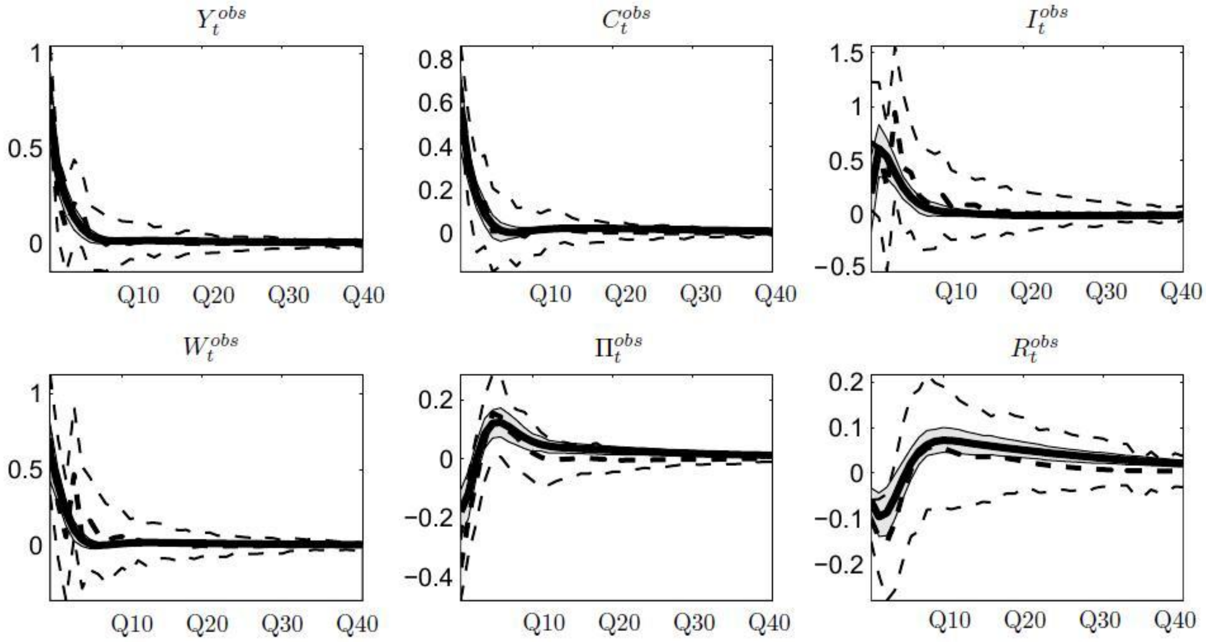
Note: bold solid line - IRF from the DSGE, grey area - 90% interval for the DSGE, bold dashed line - IRF from the DSGE-VAR, dashed lines - 90% bands for the DSGE-VAR.

the estimated DSGE-VAR not only qualitatively but also quantitatively. In both models the reactions of output, consumption and wage growth rates to the technology shock are instantaneous and rather temporary, with all variables smoothly returning to their steady states after approximately 6 to 8 quarters. The behaviour of investment growth rate, inflation and interest rate is more persistent and complex, following a hump-shape pattern. The indications of both models are alike in the case of the two latter variables, suggesting that they are virtually permanently increased by the technology shock. The responses of the investment growth rate to such a disturbance are also similar, however the DSGE-VAR suggests that in this case the impact of a technology shock is more prolonged. Compared to the IRFs with respect to the monetary policy shock, the reactions suggested by the DSGE model for the technology disturbance are slightly less precise, nevertheless they still fit the 90% intervals delivered by the DSGE-VAR model.

The explanation for such an inconsistent performance of both models, lack of coherence in the case of the monetary policy shock and accordance with respect to the technology disturbance, may be twofold. Firstly, such a result clearly indicates the misspecification of the theoretical model, which seems not to be able to track the monetary data satisfactorily enough. It seems that the number of transmission channels and introduced rigidities is insufficient to fairly explain the observed phenomena. However, taking into consideration good outcomes in terms of the remaining observables (see: Figures D.4 - D.7), the possible reason for underperformance of the DSGE model only in one dimension may lie in the data. It can be seen in the Figure D.1 that the WIBOR time series is clearly nonstationary; this feature remains even after detrending of the interest rate with the inflation target data[1]. High values of the interest rate at the beginning of the sample can be attributed to the transition of the Polish economy in the middle of

[1]The Augmented Dickey-Fuller Test for WIBOR time series yields the p-value of 0.026, therefore the unit root null hypothesis cannot be rejected at the 5% significance level.

Figure 5.2: Bayesian Impulse Response Function - technology shock.



Note: bold solid line - IRF from the DSGE, grey area - 90% interval for the DSGE,
bold dashed line - IRF from the DSGE-VAR, dashed lines - 90% bands for the DSGE-VAR.

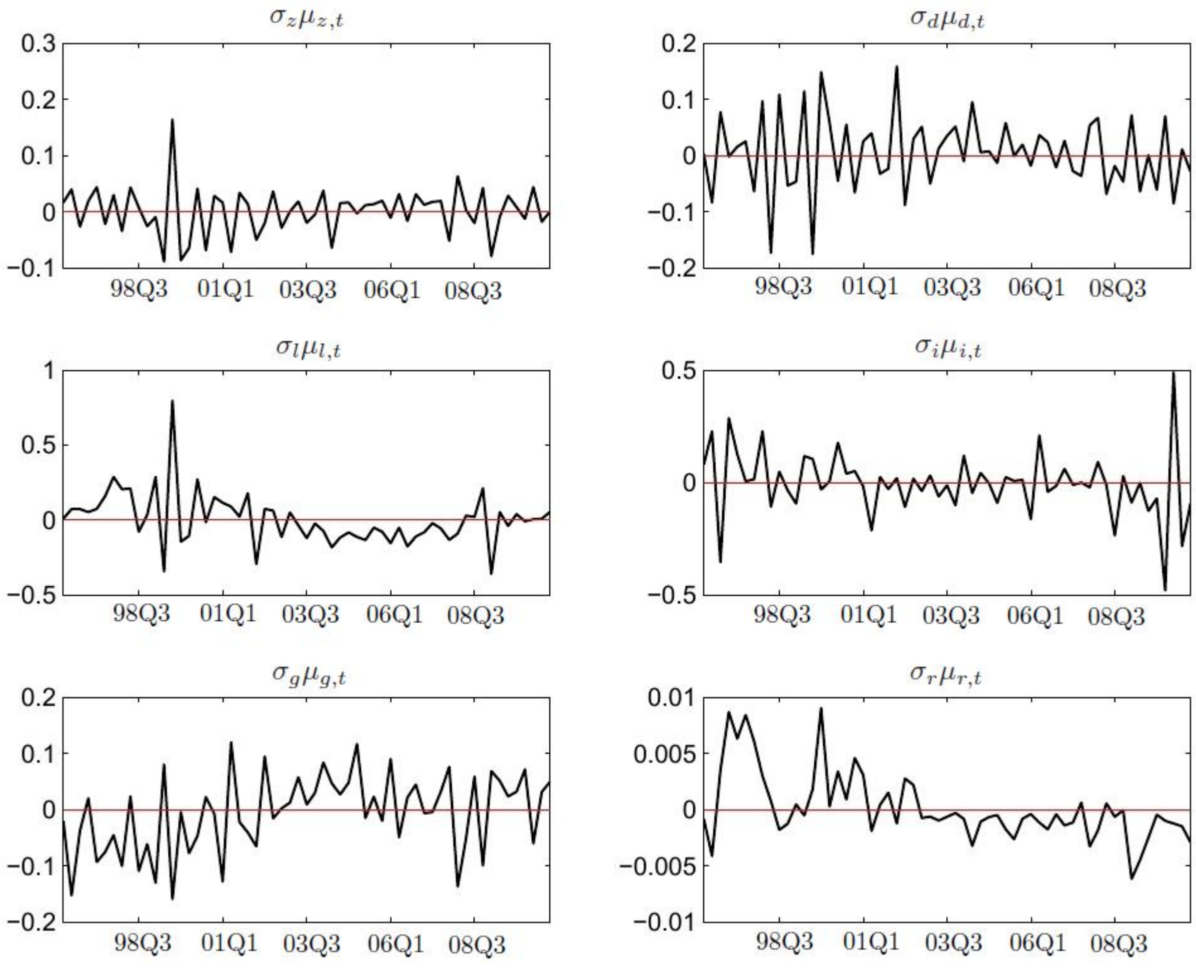
the 1990s. Assuming that the latter explanation is at least partly plausible, it may serve as justification for the DSGE model to some extend.

5.2 Stochastic disturbances

In order to examine the model empirical performance, it might be of interest to analyse the time series of the historical stochastic disturbances, which are delivered via the Kalman smoother. The Figure 5.3 presents smoothed shocks, i.e. a reconstruction of unobserved realisations of stochastic processes, computed using all the information contained in the sample, given the model structure.

It is difficult to identify the stochastic processes on the basis of the graphs below alone, yet the plots may still be useful, giving some insight into the nature of shocks hitting the Polish economy. Firstly, it seems that the volatility of disturbances lessens in the second half of the sample, which is intuitively a plausible reasoning, given turmoils present in the Polish economy in the 1990s. Secondly, the highest magnitude is revealed by the investment and labour shocks, while the technology and consumption disturbances are relatively mild. Such an observation is in line with the previously noted remarks, e.g. concerning considerable degree of habit formation in Poland. Thirdly, monetary policy shocks in the first half of the sample clearly exceed those from the second half, confirming the presumption regarding the nonstationarity of the WIBOR time series.

Figure 5.3: Smoothed (two-sided) estimates of the unobserved shocks to the transformed variables



Chapter 6

Conclusions

The main objective of this research was to model the Polish economy employing a hybrid DSGE-VAR approach. The core of the described procedure is to estimate the VAR in a Bayesian fashion, with the priors derived from a DSGE model. The specification of the latter is very close to the one of Del Negro et al. (2007), who augmented the model developed by Smets and Wouters (2003), in particular, by introducing a unit root into it, which facilitates the estimation with unfiltered data. Overall, this class of medium size DSGE models features a number of real and nominal rigidities, which allow to capture the persistence observed in the data, and has become a benchmark in the macroeconomic modelling. Similarly, the application of Bayesian techniques to model estimation is nowadays the state of the art, since it enables taking a more in-depth insight into the functioning of the economy compared to the frequentists' approach.

This paper appears to be the first attempt to implement the DSGE-VAR procedure to the Polish macroeconomic data, hence the applied DSGE model is clearly oversimplified. Nevertheless, the obtained results are fairly satisfactory. Most of the DSGE model parameters were identified and the obtained estimated are roughly in line with the previous studies of the Polish economy. Moreover, in spite of its misspecifiactions, the DSGE model seems to be a useful source of information for the VAR estimation. In terms of the generated IRFs, the DSGE-VAR model performance is inconsistent: despite good outcomes with respect to a technology disturbance, the model fails to provide plausible reactions to a monetary policy shock.

Since forecasting stays in the centre of attention of macroeconomic policy making, it would be of interest to employ the estimated DSGE-VAR model to generate future paths for macroeconomic series. This task, however, remains left for the prospective research. Moreover, the future work could extend the underlying DSGE model either slightly, by introducing additional frictions, or, more desirably, to a greater extend by opening the economy. Eventually, it would be worth comparing several competing model specifications, e.g. concerning monetary policy rules, and examine their empirical performance.

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Appendix A

Model parameters and exogenous processes

Table A.1: Structural parameters

Symbol	Interpretation
α	share of capital in output
β	household's discount factor
h	degree of habit persistence
φ	inverse of the Frisch elasticity of labour supply
δ	depreciation rate
S''	curvature of the investment cost function at the steady state
ϕ_w	substitution elasticity among labour varieties
ϕ_p	substitution elasticity among good varieties
θ_w	fraction of households unable to reoptimise wages
θ_p	fraction of firms unable to reoptimise prices
κ_w	degree of wage indexation to past inflation
κ_p	degree of price indexation to past inflation
κ_r	degree of interest rate smoothing
ϕ_π	central bank's weight on inflation gap
ϕ_y	central bank's weight on output gap
γ	technology growth rate

Table A.2: Stochastic Processes

Symbol	Law of motion	Interpretation
z_t	$z_t = (1-\rho_z)\gamma + \rho_z z_{t-1} + \sigma_z \mu_{z,t}$	growth rate of technology
$\varepsilon_{d,t}$	$\log \varepsilon_{d,t} = \rho_d \log \varepsilon_{d,t-1} + \sigma_d \mu_{d,t}$	intertemporal preferences
$\varepsilon_{l,t}$	$\log \varepsilon_{l,t} = \rho_l \log \varepsilon_{l,t-1} + \sigma_l \mu_{l,t}$	labour supply
$\varepsilon_{m,t}$	$\log \varepsilon_{m,t} = \rho_m \log \varepsilon_{m,t-1} + \sigma_m \mu_{m,t}$	preferences to money holdings
$\varepsilon_{i,t}$	$\log \varepsilon_{i,t} = \rho_i \log \varepsilon_{i,t-1} + \sigma_i \mu_{i,t}$	investment price
$\varepsilon_{g,t}$	$\log \varepsilon_{g,t} = \rho_g \log \varepsilon_{g,t-1} + \sigma_g \mu_{g,t}$	government spending
$\varepsilon_{r,t}$	$\varepsilon_{r,t} = \sigma_r \mu_{r,t}$	monetary policy rule

$\forall \iota \in \{z, d, l, m, i, g, r\} \quad \mu_{\iota,t} \overset{iid}{\sim} \mathcal{N}(0,1)$

Appendix B

Model equations

B.1 Equilibrium conditions

The list of the model equations, their stationarised and log-linearised forms, together with corresponding steady states:

a) The household's first order conditions:

- Marginal utility of consumption:

$$\text{foc: } \Lambda_{c,t}P_t = \frac{\varepsilon_{d,t}}{C_t - hC_{t-1}} - \beta h \mathbb{E}_t \left\{ \frac{\varepsilon_{d,t+1}}{C_{t+1} - hC_t} \right\}$$

$$\text{stationarised: } \lambda_{c,t} = \frac{\varepsilon_{d,t}}{c_t - h\frac{c_{t-1}}{z_t^*}} - \beta h \mathbb{E}_t \left\{ \frac{1}{z_{t+1}^*} \frac{\varepsilon_{d,t+1}}{c_{t+1} - h\frac{c_t}{z_{t+1}^*}} \right\}$$

$$\text{steady state: } \lambda_c = \frac{z^* - \beta h}{c(z^* - h)}$$

$$\begin{aligned}\text{log-linearised: } (z^* - \beta h)(z^* - h)\hat{\lambda}_{c,t} = {} & z^* h \hat{c}_{t-1} - \left((z^*)^2 + \beta h^2\right) \hat{c}_t + \beta z^* h \mathbb{E}_t\{\hat{c}_{t+1}\} \\ & - z^* h \hat{z}_t^* + \beta z^* h \mathbb{E}_t\{\hat{z}_{t+1}^*\} \\ & + z^*(z^* - h)\hat{\varepsilon}_{d,t} - \beta h (z^* - h)\mathbb{E}_t\{\hat{\varepsilon}_{d,t+1}\}\end{aligned}$$

- Consumption Euler equation:

$$\text{foc: } R_t = \frac{1}{\beta}\mathbb{E}_t \left\{ \frac{\Lambda_{c,t}}{\Lambda_{c,t+1}} \right\}$$

$$\text{stationarised: } R_t = \frac{1}{\beta}\mathbb{E}_t \left\{ \frac{\lambda_{c,t}}{\lambda_{c,t+1}} \pi_{t+1} z_{t+1}^* \right\}$$

$$\text{steady state: } R = \frac{z^*}{\beta}$$

$$\text{log-linearised: } \hat{R}_t = \hat{\lambda}_{c,t} - \mathbb{E}_t\{\hat{\lambda}_{c,t+1}\} + \mathbb{E}_t\{\hat{\pi}_{t+1}\} + \mathbb{E}_t\{\hat{z}_{t+1}^*\}$$

- Tobin-Q equation:

$$\text{foc: } Q_t = \mathbb{E}_t \left\{ (1-\delta)\frac{Q_{t+1}}{R_t}\frac{P_{t+1}}{P_t} + \frac{R^k_{t+1}}{R_t P_t} \right\},$$

$$\text{stationarised: } Q_t = \frac{1}{R_t}\mathbb{E}_t \left\{ \pi_{t+1}\left((1-\delta)Q_{t+1} + r^k_{t+1}\right) \right\}$$

$$\text{steady state: } Q = \frac{r^k}{R-(1-\delta)}$$

$$\text{log-linearised: } \hat{Q}_t = -\hat{R}_t + \mathbb{E}_t\{\hat{\pi}_{t+1}\} + \frac{1-\delta}{R}\mathbb{E}_t\{\hat{Q}_{t+1}\} + \left(1 - \frac{1-\delta}{R}\right)\mathbb{E}_t\{\hat{r}^k_{t+1}\}$$

- Investment equation:

$$\text{foc: } 1 - \varepsilon_{i,t}Q_t \left[1 - S\left(\frac{I_t}{I_{t-1}}\right) - \frac{I_t}{I_{t-1}}S'\left(\frac{I_t}{I_{t-1}}\right)\right] = \mathbb{E}_t \left\{ \varepsilon_{i,t+1}\frac{Q_{t+1}}{R_t}\frac{P_{t+1}}{P_t}\left(\frac{I_{t+1}}{I_t}\right)^2 S'\left(\frac{I_{t+1}}{I_t}\right) \right\}$$

$$\text{stationarised: } 1 - \varepsilon_{i,t}Q_t \left[1 - S\left(\frac{i_t}{i_{t-1}}z^*_t\right) - \frac{i_t}{i_{t-1}}z^*_t S'\left(\frac{i_t}{i_{t-1}}z^*_t\right)\right]$$

$$= \mathbb{E}_t \left\{ \varepsilon_{i,t+1}\frac{Q_{t+1}}{R_t}\pi_{t+1}\left(\frac{i_{t+1}}{i_t}z^*_{t+1}\right)^2 S'\left(\frac{i_{t+1}}{i_t}z^*_{t+1}\right) \right\}$$

$$\text{steady state: } Q = 1$$

$$\text{log-linearised: } \hat{i}_t = \frac{1}{1+\beta}\hat{i}_{t-1} + \frac{\beta}{1+\beta}\mathbb{E}_t\{\hat{i}_{t+1}\} - \frac{1}{1+\beta}\hat{z}^*_t - \frac{\beta}{1+\beta}\mathbb{E}_t\{\hat{z}^*_{t+1}\} + \frac{\hat{Q}_t + \hat{\varepsilon}_{i,t}}{(1+\beta)(z^*)^2 S''}$$

- Wage equation:

$$\text{foc: } \mathbb{E}_t \sum_{s=0}^{\infty}(\theta_w\beta)^s L_{t+s}\left\{ \frac{\tilde{W}_t}{P_{t+s}}\left(\frac{P_{t+s-1}}{P_{t-1}}\right)^{\kappa_w}(U_{c,t+s} - \beta h U_{c,t+s+1}) - \frac{\phi_w}{\phi_w - 1}U_{l,t+s} \right\} = 0$$

$$\text{or, equivalently: } \mathbb{E}_t \sum_{s=0}^{\infty}(\theta_w\beta)^s L_{t+s}\left\{ \Lambda_{c,t+s}\tilde{W}_t\left(\frac{P_{t+s-1}}{P_{t-1}}\right)^{\kappa_w} - \frac{\phi_w}{\phi_w - 1}U_{l,t+s} \right\} = 0$$

$$\text{stationarised: } \mathbb{E}_t \sum_{s=0}^{\infty}(\theta_w\beta)^s L_{t+s}\left\{ \lambda_{c,t+s}\tilde{w}_t\left(\frac{P_{t+s-1}}{P_{t-1}}\right)^{\kappa_w} - \frac{\phi_w}{\phi_w - 1}U_{l,t+s} \right\} = 0$$

$$\text{steady state: } \lambda_c\tilde{w} = \frac{\phi_w}{\phi_w - 1}U_l$$

$$\text{log-linearised: } \mathbb{E}_t \sum_{s=0}^{\infty}(\theta_w\beta)^s \left\{ \hat{\lambda}_{c,t+s} + \hat{\tilde{w}}_t + \kappa_w(\hat{P}_{t+s-1} - \hat{P}_{t-1}) - \hat{U}_{l,t+s} \right\} = 0$$

$$\text{where: } U_l = -L^{\varphi}, \quad \hat{U}_{l,t} = \hat{\varepsilon_{d,t}} + \hat{\varepsilon}_{l,t} + \varphi\hat{L}_t$$

b) The intermediate firm's first order conditions:

- Real cost of capital:

$$\text{foc: } \frac{K_t}{L_t} = \frac{\alpha}{1-\alpha}\frac{W_t}{R_t^k}$$
$$\text{stationarised: } \frac{k_t}{z_t^* L_t} = \frac{\alpha}{1-\alpha}\frac{w_t}{r_t^k}$$
$$\text{steady state: } \frac{k}{z^* L} = \frac{\alpha}{1-\alpha}\frac{w}{r^k}$$
$$\text{log-linearised: } \hat{r}_t^k = \hat{L}_t + \hat{w}_t + \hat{z}_t^* - \hat{k}_t$$

- Marginal cost:

$$\text{foc: } MC_t = \left(\frac{W_t}{(1-\alpha)Z_t}\right)^{1-\alpha}\left(\frac{R_t^k}{\alpha}\right)^{\alpha},$$
$$\text{stationarised: } mc_t = \left(\frac{w_t}{(1-\alpha)Z_t}\right)^{1-\alpha}\left(\frac{r_t^k}{\alpha}\right)^{\alpha},$$
$$\text{steady state: } mc = \left(\frac{w}{1-\alpha}\right)^{1-\alpha}\left(\frac{r^k}{\alpha}\right)^{\alpha}$$
$$\text{log-linearised: } \hat{mc}_t = (1-\alpha)\hat{w}_t + \alpha\hat{r}_t^k,$$

- Pricing equation:

$$\text{foc: } \mathbb{E}_t\sum_{s=0}^{\infty}\theta_p^s \Xi_{t+s} Y_{t+s}\left[\tilde{P}_t\left(\frac{P_{t+s-1}}{P_{t-1}}\right)^{\kappa_p} - \frac{\phi_p}{\phi_p - 1}MC_{t+s}\right] = 0$$
$$\text{stationarised: } \mathbb{E}_t\sum_{s=0}^{\infty}(\theta_p\beta)^s \lambda_{c,t+s} y_{t+s}\left[\frac{\tilde{P}_t}{P_{t+s}}\left(\frac{P_{t+s-1}}{P_{t-1}}\right)^{\kappa_p} - \frac{\phi_p}{\phi_p - 1}mc_{t+s}\right] = 0$$
$$\text{steady state: } mc = \frac{\phi_p - 1}{\phi_p}$$
$$\text{log-linearised: } \mathbb{E}_t\sum_{s=0}^{\infty}(\theta_p\beta)^s\left(\hat{\tilde{P}}_t - \hat{P}_{t+s} + \kappa_p(\hat{P}_{t+s-1} - \hat{P}_{t-1}) - \hat{mc}_{t+s}\right) = 0$$

c) Technological constraints:

- Capital accumulation:

$$K_{t+1} = (1-\delta)K_t + \varepsilon_{i,t}\left(1 - S\left(\frac{I_t}{I_{t-1}}\right)\right)I_t$$
$$\text{stationarised: } k_{t+1}z_{t+1}^* = (1-\delta)k_t + \varepsilon_{i,t}\left(1 - S\left(\frac{i_t}{i_{t-1}}z_t^*\right)\right)i_t$$
$$\text{steady state: } i = [\, z^* - (1-\delta)]k$$
$$\text{log-linearised: } \hat{k}_{t+1} = \frac{1-\delta}{z^*}\hat{k}_t + \left(1 - \frac{1-\delta}{z^*}\right)\left(\hat{i}_t + \hat{\varepsilon}_{i,t}\right) - \hat{z}_{t+1}^*$$

- Production function

$$Y_t = Z_t^{1-\alpha} L_t^{1-\alpha} K_t^{\alpha} - Z_t \Phi$$

$$\text{stationarised: } y_t = L_t^{1-\alpha} \left(\frac{k_t}{z_t^*}\right)^{\alpha} - \Phi$$

$$\text{steady state: } y = L^{1-\alpha} \left(\frac{k}{z^*}\right)^{\alpha}$$

$$\text{log-linearised: } \hat{y}_t = (1-\alpha)\hat{L}_t + \alpha\left(\hat{k}_t - \hat{z}_t^*\right)$$

d) Wage and price indices:

- Price index:

$$P_t = \left[(1-\theta_p)\tilde{P}_t^{1-\phi_p} + \theta_p(\pi_{t-1}^{\kappa_p} P_{t-1})^{1-\phi_p}\right]^{\frac{1}{1-\phi_p}}$$

$$\text{steady state: } P = \tilde{P}$$

$$\text{log-linearised: } \hat{P}_t = (1-\theta_p)\hat{\tilde{P}}_t + \theta_p(\hat{P}_{t-1} + \kappa_p \hat{\pi}_{t-1})$$

- Wage index:

$$W_t = \left[(1-\theta_w)\tilde{W}_t^{1-\phi_w} + \theta_w(\pi_{t-1}^{\kappa_w} W_{t-1})^{1-\phi_w}\right]^{\frac{1}{1-\phi_w}}$$

$$\text{stationarised: } w_t^{1-\phi_w} = (1-\theta_w)\tilde{w}_t^{1-\phi_w} + \theta_w\left(\frac{\pi_{t-1}^{\kappa_w}}{\pi_t z_t^*} w_{t-1}\right)^{1-\phi_w}$$

$$\text{steady state: } w = \tilde{w}$$

$$\text{log-linearised: } \hat{w}_t = (1-\theta_w)\hat{\tilde{w}}_t + \theta_w(\hat{w}_{t-1} + \kappa_w \hat{\pi}_{t-1} - \hat{\pi}_t - \hat{z}_t^*)$$

$$\hat{w}_t = (1-\theta_w)\hat{\tilde{w}}_t + \theta_w(\hat{w}_{t-1} + \kappa_w \hat{\pi}_{t-1} - \hat{\pi}_t - \gamma \hat{z}_t)$$

e) Taylor rule:

$$\frac{R_t}{R} = \left(\frac{R_{t-1}}{R}\right)^{\kappa_r} \left[\left(\frac{\pi_t}{\pi}\right)^{\phi_\pi} \left(\frac{Y_t}{Y_t^*}\right)^{\phi_y}\right]^{1-\kappa_r} \exp\left(\varepsilon_{r,t}\right)$$

$$\text{stationarised: } \frac{R_t}{R} = \left(\frac{R_{t-1}}{R}\right)^{\kappa_r} \left[\left(\frac{\pi_t}{\pi}\right)^{\phi_\pi} \left(\frac{y_t}{y}\right)^{\phi_y}\right]^{1-\kappa_r} \exp\left(\varepsilon_{r,t}\right)$$

$$\text{log-linearised: } \hat{R}_t = \kappa_r \hat{R}_{t-1} + (1-\kappa_r)(\phi_\pi \hat{\pi}_t + \phi_y \hat{y}_t) + \varepsilon_{r,t}$$

f) Market clearing condition:

$$Y_t = C_t + I_t + G_t$$

$$\text{stationarised: } y_t = c_t + i_t + g_t$$

$$\text{steady state: } y = c + i + g$$

$$\text{log-linearised: } \hat{y}_t = \frac{c}{y}\hat{c}_t + \frac{i}{y}\hat{i}_t + \frac{g}{y}\hat{g}_t$$

$$\text{where: } \hat{g}_t = \hat{\varepsilon}_{g,t}$$

B.2 Derivation of the Phillips curve and the wage equation

Below, to spare on notation, the rational expectation operators have been omitted and the variables dated $t+1$ refer to the rational expectation of those variables.

Phillips curve:

$$\sum_{s=0}^{\infty}(\theta_p\beta)^s\left(\hat{\hat{P}}_t - \hat{P}_{t+s} + \kappa_p(\hat{P}_{t+s-1} - \hat{P}_{t-1}) - \hat{mc}_{t+s}\right) = 0 \quad \text{implies:}$$

$$\left.\begin{array}{ll} \text{for t:} & \frac{1}{1-\beta\theta_p}\left(\hat{\hat{P}}_t - \kappa_p\hat{P}_{t-1}\right) = \sum_{s=0}^{\infty}(\theta_p\beta)^s\left(\hat{P}_{t+s} - \kappa_p\hat{P}_{t+s-1} + \hat{mc}_{t+s}\right) \\ \text{for t+1:} & \frac{1}{1-\beta\theta_p}\left(\hat{\hat{P}}_{t+1} - \kappa_p\hat{P}_t\right) = \sum_{s=0}^{\infty}(\theta_p\beta)^s\left(\hat{P}_{t+1+s} - \kappa_p\hat{P}_{t+s} + \hat{mc}_{t+1+s}\right) \end{array}\right\} \Rightarrow$$

$$\frac{1}{1-\beta\theta_p}\left(\hat{\hat{P}}_t - \kappa_p\hat{P}_{t-1}\right) = \left(\hat{P}_t - \kappa_p\hat{P}_{t-1} + \hat{mc}_t\right) + \beta\theta_p\cdot\frac{1}{1-\beta\theta_p}\left(\hat{\hat{P}}_{t+1} - \kappa_p\hat{P}_t\right)$$

$$\begin{aligned} \hat{\hat{P}}_t - \kappa_p\hat{P}_{t-1} &= (1-\beta\theta_p)\left(\hat{P}_t - \kappa_p\hat{P}_{t-1} + \hat{mc}_t\right) + \beta\theta_p\left(\hat{\hat{P}}_{t+1} - \kappa_p\hat{P}_t\right) \\ \hat{\hat{P}}_t - \beta\theta_p\hat{\hat{P}}_{t+1} &= (1-\beta\theta_p)\left(\hat{P}_t - \kappa_p\hat{P}_{t-1} + \hat{mc}_t\right) + \kappa_p\left(\hat{P}_{t-1} - \beta\theta_p\hat{P}_t\right) \\ \hat{\hat{P}}_t - \beta\theta_p\hat{\hat{P}}_{t+1} &= (1-\beta\theta_p)\hat{P}_t - \beta\theta_p\kappa_p(\hat{P}_t - \hat{P}_{t-1}) + (1-\beta\theta_p)\hat{mc}_t \\ \underbrace{\hat{\hat{P}}_t - \beta\theta_p\hat{\hat{P}}_{t+1} - (1-\beta\theta_p)\hat{P}_t}_{LHS} &= -\beta\theta_p\kappa_p\hat{pi}_t + (1-\beta\theta_p)\hat{mc}_t \end{aligned}$$

$$\begin{aligned} \hat{P}_t &= (1-\theta_p)\hat{\hat{P}}_t + \theta_p(\hat{P}_{t-1} + \kappa_p\hat{\pi}_{t-1}) \quad \text{implies:} \\ \hat{\hat{P}}_t &= \frac{1}{1-\theta_p}\left(\hat{P}_t - \theta_p(\hat{P}_{t-1} + \kappa_p\hat{\pi}_{t-1})\right) \quad \text{thus:} \end{aligned}$$

$$\begin{aligned} LHS &= \hat{\hat{P}}_t - \beta\theta_p\hat{\hat{P}}_{t+1} - (1-\beta\theta_p)\hat{P}_t \\ &= \frac{1}{1-\theta_p}\left[\hat{P}_t - \beta\theta_p\hat{P}_{t+1} - \theta_p(\hat{P}_{t-1} - \beta\theta_p\hat{P}_t) - \theta_p\kappa_p(\hat{\pi}_{t-1} - \beta\theta_p\hat{\pi}_t)\right] - (1-\beta\theta_p)\hat{P}_t \\ &= \frac{1}{1-\theta_p}\left[-\beta\theta_p(\hat{P}_{t+1} - \hat{P}_t) + \theta_p(\hat{P}_t - \hat{P}_{t-1}) - \theta_p\kappa_p(\hat{\pi}_{t-1} - \beta\theta_p\hat{\pi}_t)\right] \\ &= \frac{\theta_p}{1-\theta_p}\left[-\beta\hat{\pi}_{t+1} + (1+\beta\theta_p\kappa_p)\hat{\pi}_t - \kappa_p\hat{\pi}_{t-1}\right] \end{aligned}$$

$$LHS + \beta\theta_p\kappa_p\hat{\pi}_t = \frac{\theta_p}{1-\theta_p}\Big[-\beta\hat{\pi}_{t+1} + (1+\beta\kappa_p)\hat{\pi}_t - \kappa_p\hat{\pi}_{t-1}\Big] = (1-\beta\theta_p\hat{mc}_t)$$

$$\hat{\pi}_t = \frac{\beta}{1+\beta\kappa_p}\hat{\pi}_{t+1} + \frac{\kappa_p}{1+\beta\kappa_p}\hat{\pi}_{t-1} + \frac{1-\theta_p}{\theta_p}\frac{1-\beta\theta_p}{1+\beta\kappa_p}\hat{mc}_t$$

Since $\hat{mc}_t = (1-\alpha)\hat{w}_t + \alpha\hat{r}_t^k$, finally one obtains:

$$\hat{\pi}_t = \frac{\beta}{1+\beta\kappa_p}\hat{\pi}_{t+1} + \frac{\kappa_p}{1+\beta\kappa_p}\hat{\pi}_{t-1} + \frac{1-\theta_p}{\theta_p}\frac{1-\beta\theta_p}{1+\beta\kappa_p}\Big((1-\alpha)\hat{w}_t + \alpha\hat{r}_t^k\Big)$$

Wage equation:

$$\sum_{s=0}^{\infty}(\theta_w\beta)^s\left(\hat{\lambda}_{c,t+s} + \hat{\tilde{w}}_t + \kappa_w(\hat{P}_{t+s-1} - \hat{P}_{t-1}) - \hat{U}_{l,t+s}\right) = 0$$

$$\left.\begin{array}{ll} \text{for t:} & \frac{1}{1-\beta\theta_w}\left(\hat{\tilde{w}}_t - \kappa_w\hat{P}_{t-1}\right) = \sum_{s=0}^{\infty}(\theta_w\beta)^s\left(\hat{U}_{l,t+s} - \hat{\lambda}_{c,t+s} - \kappa_w\hat{P}_{t+s-1}\right) \\ \text{for t+1:} & \frac{1}{1-\beta\theta_w}\left(\hat{\tilde{w}}_{t+1} - \kappa_w\hat{P}_t\right) = \sum_{s=0}^{\infty}(\theta_w\beta)^s\left(\hat{U}_{l,t+1+s} - \hat{\lambda}_{c,t+1+s} - \kappa_w\hat{P}_{t+s}\right) \end{array}\right\} \Rightarrow$$

$$\frac{1}{1-\beta\theta_w}\left(\hat{\tilde{w}}_t - \kappa_w\hat{P}_{t-1}\right) = \left(\hat{U}_{l,t} - \hat{\lambda}_{c,t} - \kappa_w\hat{P}_{t-1}\right) + \beta\theta_w\cdot\frac{1}{1-\beta\theta_w}\left(\hat{\tilde{w}}_{t+1} - \kappa_w\hat{P}_t\right)$$

$$\hat{\tilde{w}}_t - \kappa_w\hat{P}_{t-1} = (1-\beta\theta_w)\left(\hat{U}_{l,t} - \hat{\lambda}_{c,t} - \kappa_w\hat{P}_t\right) + \beta\theta_w\left(\hat{\tilde{w}}_{t+1} - \kappa_w\hat{P}_t\right)$$

$$\hat{\tilde{w}}_t - \beta\theta_w\hat{\tilde{w}}_{t+1} = -\beta\theta_w\kappa_w(\hat{P}_t - \hat{P}_{t-1}) + (1-\beta\theta_w)\left(\hat{U}_{l,t} - \hat{\lambda}_{c,t}\right)$$

$$\underbrace{\hat{\tilde{w}}_t - \beta\theta_w\hat{\tilde{w}}_{t+1}}_{LHS} = -\beta\theta_w\kappa_w\hat{\pi}_t + (1-\beta\theta_w)\left(\hat{U}_{l,t} - \hat{\lambda}_{c,t}\right)$$

$$\hat{w}_t = (1-\theta_w)\hat{\tilde{w}}_t + \theta_w(\hat{w}_{t-1} + \kappa_w\hat{\pi}_{t-1} - \hat{\pi}_t - \hat{z}_t^*) \quad \text{implies:}$$

$$\hat{\tilde{w}}_t = \frac{1}{(1-\theta_w)}\Big[\hat{w}_t - \theta_w(\hat{w}_{t-1} + \kappa_w\hat{\pi}_{t-1} - \hat{\pi}_t - \hat{z}_t^*)\Big] \quad \text{thus:}$$

$$\begin{aligned} LHS = \ & \hat{\tilde{w}}_t - \beta\theta_w\hat{\tilde{w}}_{t+1} \\ = \ & \frac{1}{1-\theta_w}\Big[\hat{w}_t - \beta\theta_w\hat{w}_{t+1} \\ & - \theta_w(\hat{w}_{t-1} - \beta\theta_w\hat{w}_t) - \theta_w(\kappa_w(\hat{\pi}_{t-1} - \beta\theta_w\hat{\pi}_t) - (\hat{\pi}_t - \beta\theta_w\hat{\pi}_{t+1}) - (\hat{z}_t^* - \beta\theta_w\hat{z}_{t+1}^*))\Big] \\ = \ & \frac{1}{1-\theta_w}\Big[(1+\beta(\theta_w)^2)\hat{w}_t - \beta\theta_w\hat{w}_{t+1} - \theta_w\hat{w}_{t-1} \\ & + \theta_w[(1+\beta\theta_w\kappa_w)\hat{\pi}_t - \beta\theta_w\hat{\pi}_{t+1} - \kappa_w\hat{\pi}_{t-1}] + \theta_w[\hat{z}_t^* - \beta\theta_w\hat{z}_{t+1}^*]\Big] \end{aligned}$$

$$\begin{aligned} LHS + \beta\theta_w\kappa_w\hat{\pi}_t = \ & \frac{1}{1-\theta_w}\big\{(1+\beta(\theta_w)^2)\hat{w}_t - \beta\theta_w\hat{w}_{t+1} - \theta_w\hat{w}_{t-1} + \theta_w[(1+\beta\theta_w\kappa_w)\hat{\pi}_t - \beta\theta_w\hat{\pi}_{t+1} - \kappa_w\hat{\pi}_{t-1}] \\ & + \theta_w[\hat{z}_t^* - \beta\theta_w\hat{z}_{t+1}^*]\big\} \\ = \ & (1-\beta\theta_w)\left(\hat{U}_{l,t} - \hat{\lambda}_{c,t}\right) \end{aligned}$$

Finally it can be written:

$$
\begin{aligned}
\hat{w}_t = \quad & \frac{\theta_w}{1+\beta(\theta_w)^2}\left[\beta\hat{w}_{t+1} + \hat{w}_{t-1} - (1+\beta\kappa_w)\hat{\pi}_t + \beta\theta_w\hat{\pi}_{t+1} + \kappa_w\hat{\pi}_{t-1}\right. \\
& \left. -\hat{z}_t^* + \beta\theta_w\hat{z}_{t+1}^* + \frac{1-\theta_w}{\theta_w}(1-\beta\theta_w)\left(\hat{U}_{l,t} - \hat{\lambda}_{c,t}\right)\right]
\end{aligned}
$$

Appendix C

Dynare code

```
var lam c R Q i rk L Pi w y k z
    eps_d eps_l eps_i eps_g
    GDP CONS INV WG CPI WIBOR;

varexo mu_z mu_d mu_l mu_i mu_g mu_r;

parameters alpha beta delta phi_w phi_p c_ss g_ss
           h varphi S theta_w theta_p kappa_w kappa_p kappa_r phi_Pi phi_y
           gamma_obs rho_z rho_d rho_l rho_i rho_g ;

//Calibrated parameters
alpha   = 0.33;
beta    = 0.995;
delta   = 0.025;
phi_w   = 3;
phi_p   = 3;
c_ss    = 0.582;
g_ss    = 0.2;

model(linear);

//Endogenous variables
# gamma = 1 + gamma_obs;
(gamma-beta*h)*(gamma-h)*lam = -(gamma^2+beta*h^2)*c + gamma*h*c(-1) + beta*gamma*h*c(+1)
    - gamma*h*z  + beta*gamma*z(+1) + (gamma-h)*gamma*eps_d - (gamma-h)*beta*h*eps_d(+1)
    ;
R  = lam - lam(+1) + Pi(+1) + z(+1);
Q  = - R + Pi(+1) + (1-delta)*beta/gamma*Q(+1) + (1-(1-delta)/gamma)*beta*rk(+1);
i  = 1/(1+beta)*(i(-1)-z)+ beta/(1+beta)*(i(+1)-z(+1)) + (Q+eps_i)/((1+beta)*S*(gamma^2))
    ;
rk = L + w + z - k(-1);
R  = kappa_r*R(-1) + (1-kappa_r)*(phi_Pi*Pi + phi_y*y) + mu_r;
```

```
y  = c_ss*c + (1-c_ss-g_ss)*i + g_ss*eps_g;
k  = (1-delta)/gamma*(k(-1)-z) + (1-(1-delta)/gamma)*(i+eps_i);
y  = (1-alpha)*L + alpha*(k(-1)-z);
Pi = beta/(1+beta*kappa_p)*Pi(+1) + kappa_p/(1+beta*kappa_p)*Pi(-1)
     + (1-theta_p)*(1-beta*theta_p)/theta_p/(1+beta*kappa_p)*((1-alpha)*w + alpha*rk);
w  = theta_w/(1+beta*(theta_w^2))*( beta*w(+1) + w(-1)
     - (1+beta*kappa_w)*Pi + beta*theta_w*Pi(+1) + kappa_w*Pi(-1)
     - z + beta*theta_w*z(+1) + (1-theta_w)*(1-beta*theta_w)/theta_w*(eps_d+eps_l+varphi*
         L-lam));

// Exogenous processes
z     = rho_z*z(-1) + mu_z;
eps_d = rho_d*eps_d(-1) + mu_d;
eps_l = rho_l*eps_l(-1) + mu_l;
eps_i = rho_i*eps_i(-1) + mu_i;
eps_g = rho_g*eps_g(-1) + mu_g;

// Measurement equations - quaterly data
GDP   = 100*(y-y(-1)+z+log(gamma));
CONS  = 100*(c-c(-1)+z+log(gamma));
INV   = 100*(i-i(-1)+z+log(gamma));
WG    = 100*(w-w(-1)+z+log(gamma));
CPI   = 100*Pi;
WIBOR = 100*(R+log(gamma/beta));

end;

varobs GDP CONS INV WG CPI WIBOR;

//steady;
//check;

estimated_params;

   h,                   beta_pdf,          0.70,     0.10;
   varphi,              gamma_pdf,         2.00,     0.70;
   S,                   gamma_pdf,         6.00,     1.50;

   theta_w,             beta_pdf,          0.60,     0.10;
   theta_p,             beta_pdf,          0.60,     0.10;
   kappa_w,             beta_pdf,          0.50,     0.15;
   kappa_p,             beta_pdf,          0.50,     0.15;

   kappa_r,             beta_pdf,          0.80,     0.08;
   phi_Pi,              gamma_pdf,         1.70,     0.15;
   phi_y,               gamma_pdf,         0.125,    0.065;

   gamma_obs,           gamma_pdf,         0.008,    0.0015;

```

```
  rho_z,             beta_pdf,        0.40,     0.1;
  rho_d,             beta_pdf,        0.70,     0.1;
  rho_l,             beta_pdf,        0.70,     0.1;
  rho_i,             beta_pdf,        0.70,     0.1;
  rho_g,             beta_pdf,        0.70,     0.1;

  stderr mu_z,       inv_gamma_pdf,   0.0050,   inf;
  stderr mu_d,       inv_gamma_pdf,   0.0050,   inf;
  stderr mu_l,       inv_gamma_pdf,   0.0050,   inf;
  stderr mu_i,       inv_gamma_pdf,   0.0075,   inf;
  stderr mu_g,       inv_gamma_pdf,   0.0100,   inf;
  stderr mu_r,       inv_gamma_pdf,   0.0030,   inf;

  dsge_prior_weight, uniform_pdf,,,   0,        2;

end;

estimation(datafile=data, xls_sheet=quarterly, xls_range=B2:G65,
           plot_priors=0, mode_file=dsge_v9_est_m6_mode,
           mh_replic=500000, mh_drop=0.5, mh_jscale=0.45, mode_compute=1, mode_check,
           bayesian_irf, dsge_var, first_obs=5, order=1);
```

Appendix D

Charts and figures

Figure D.1: Historical transformed variables

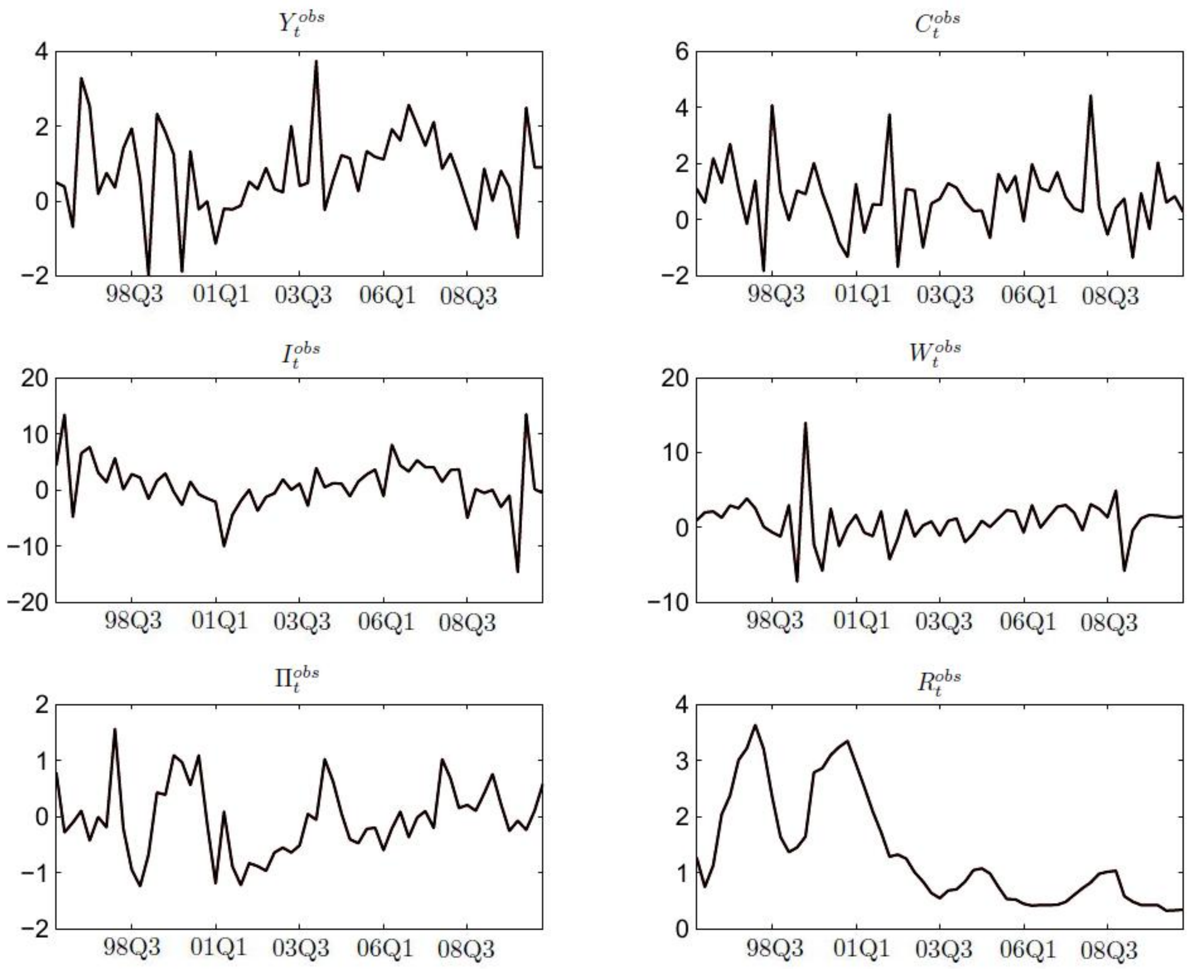
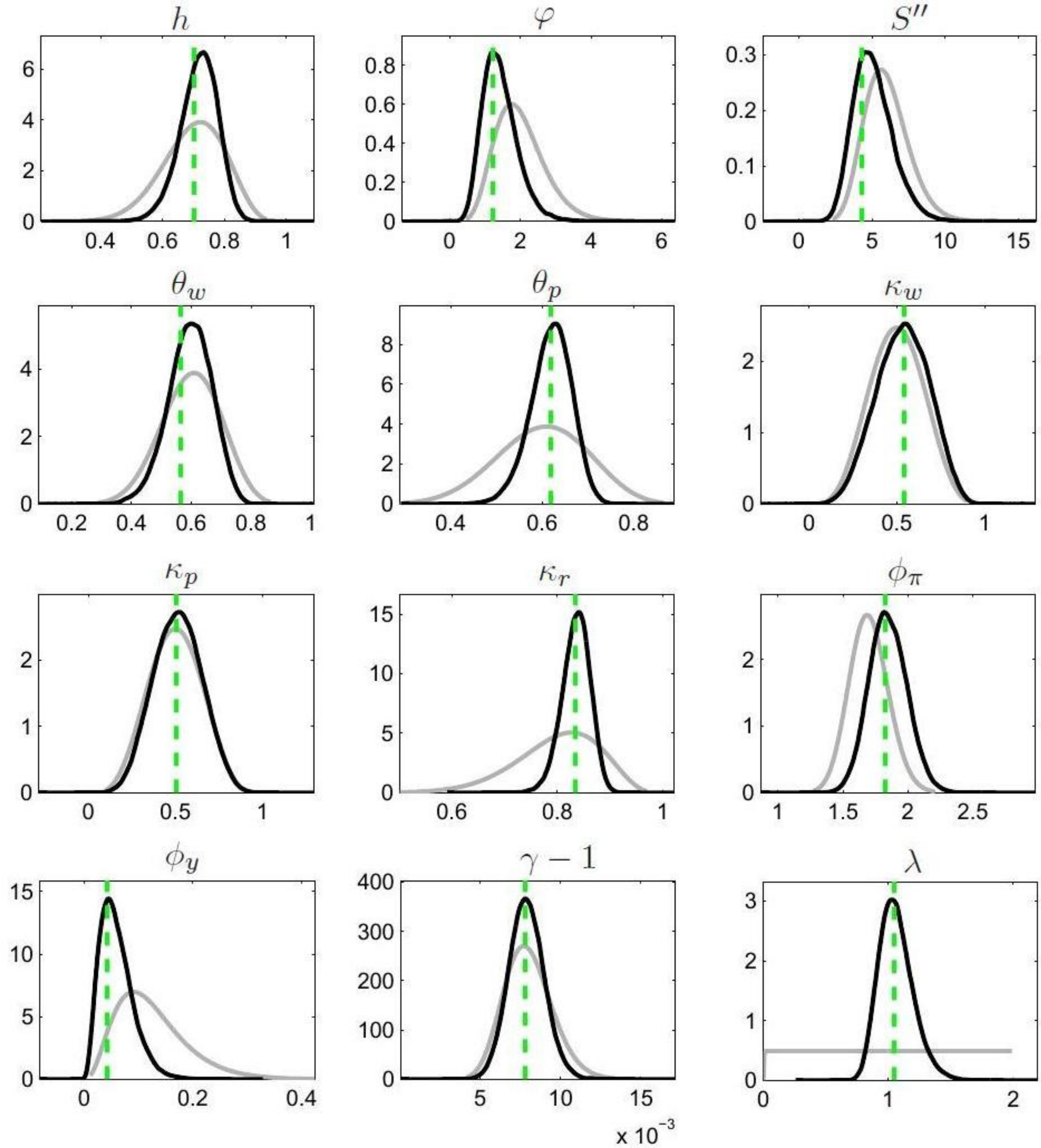
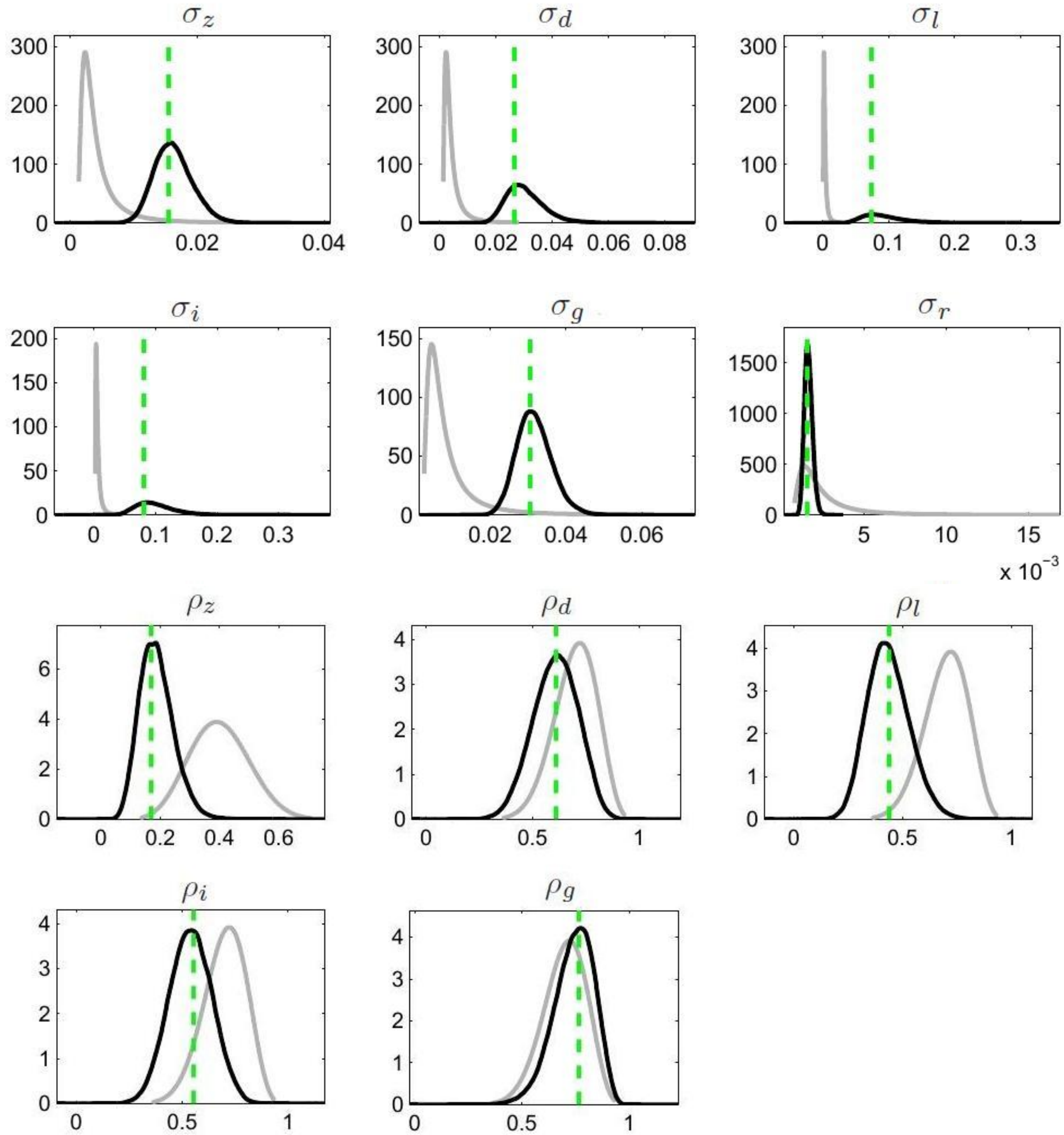


Figure D.2: Prior and posterior densities - structural parameters



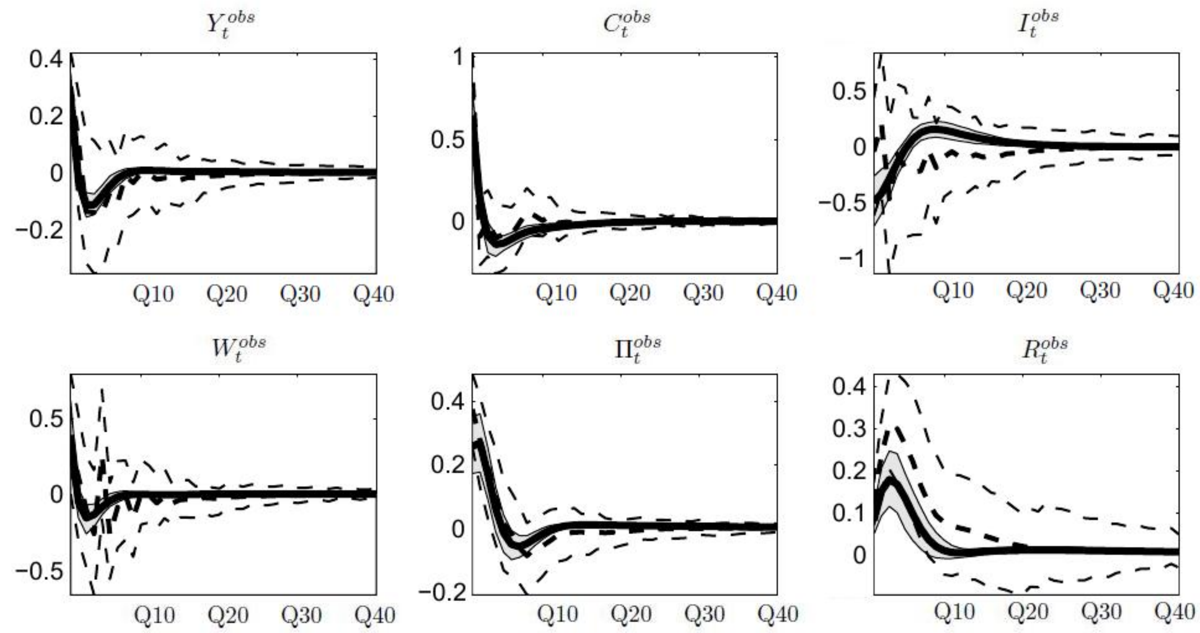
Note: grey line - prior distribution, black line - posterior distribution, vertical green dashed line - posterior mode.

Figure D.3: Prior and posterior densities - stochastic processes



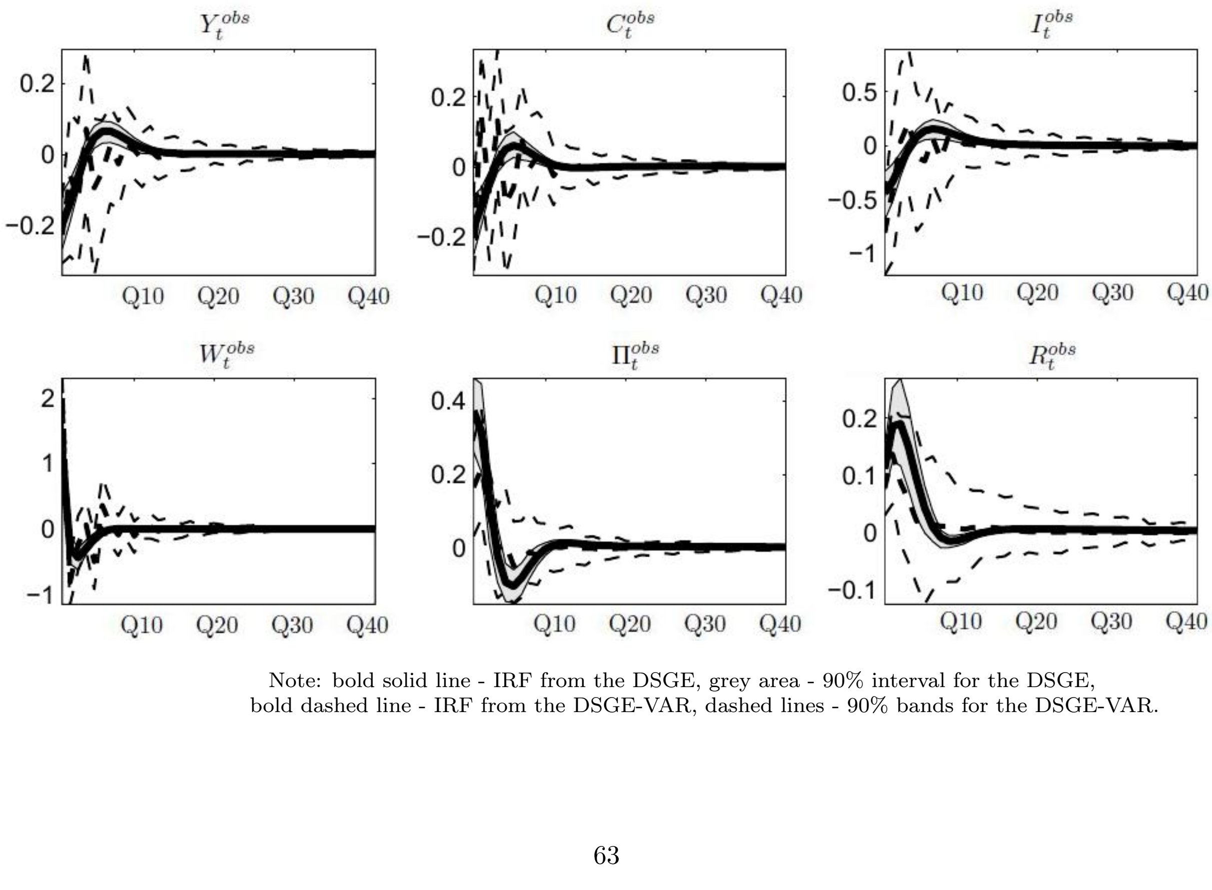
Note: grey line - prior distribution, black line - posterior distribution, vertical green dashed line - posterior mode.

Figure D.4: Bayesian Impulse Response Function - preference shock.



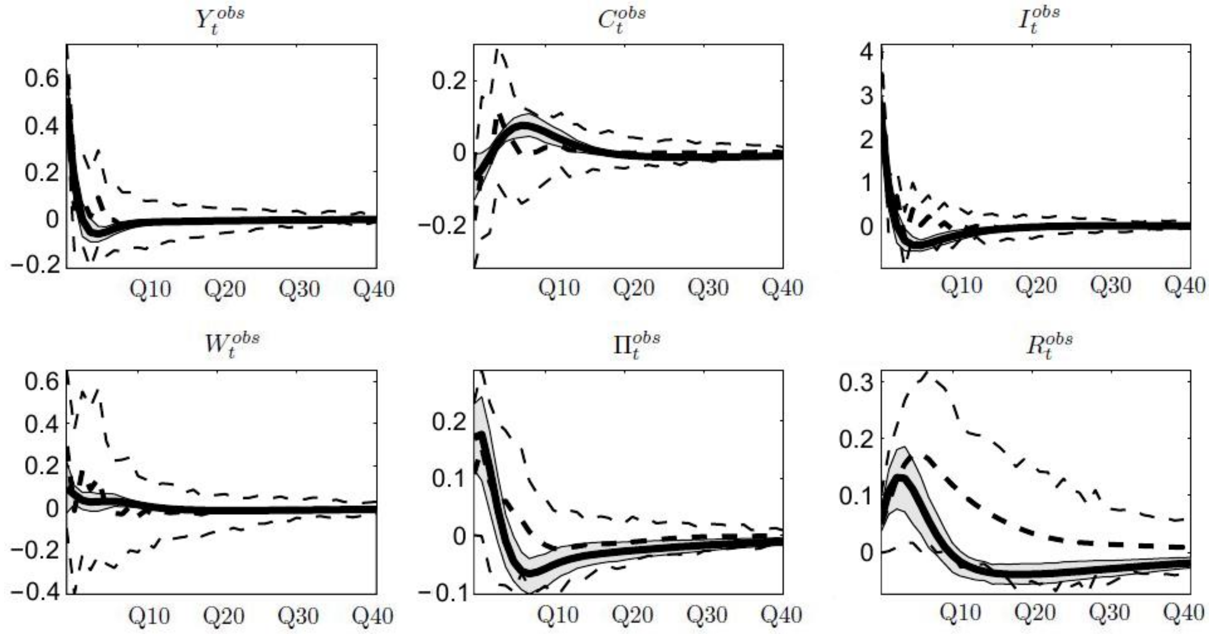
Note: bold solid line - IRF from the DSGE, grey area - 90% interval for the DSGE, bold dashed line - IRF from the DSGE-VAR, dashed lines - 90% bands for the DSGE-VAR.

Figure D.5: Bayesian Impulse Response Function - labour supply shock.



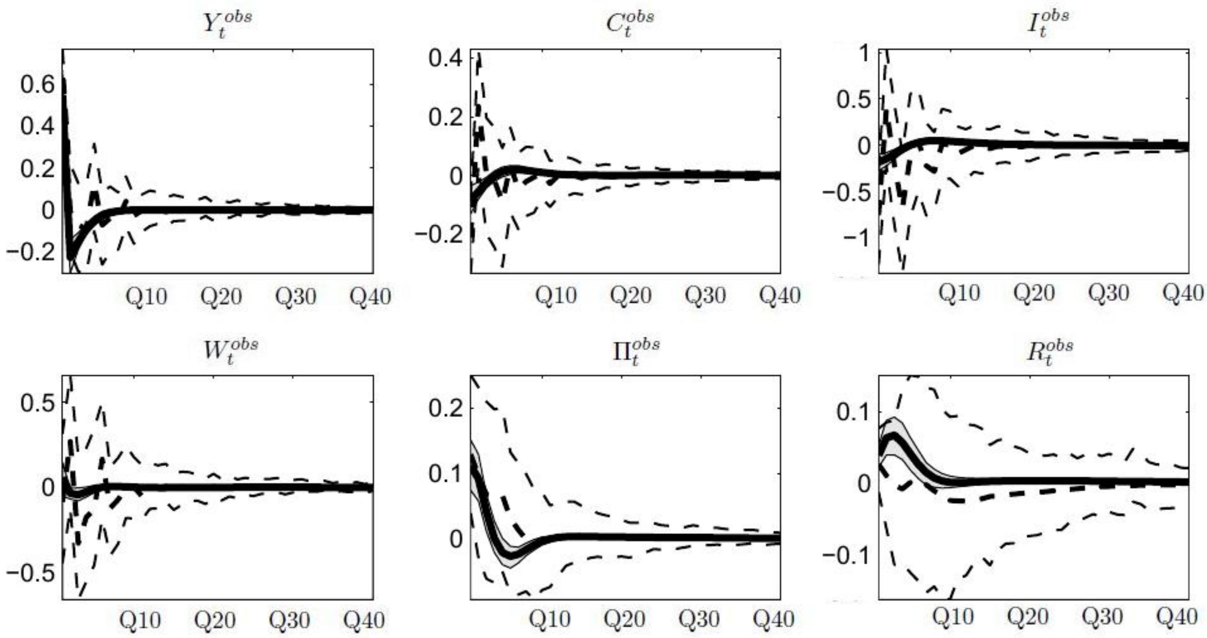
Note: bold solid line - IRF from the DSGE, grey area - 90% interval for the DSGE, bold dashed line - IRF from the DSGE-VAR, dashed lines - 90% bands for the DSGE-VAR.

Figure D.6: Bayesian Impulse Response Function - investment shock.



Note: bold solid line - IRF from the DSGE, grey area - 90% interval for the DSGE,
bold dashed line - IRF from the DSGE-VAR, dashed lines - 90% bands for the DSGE-VAR.

Figure D.7: Bayesian Impulse Response Function - government spending shock.



Note: bold solid line - IRF from the DSGE, grey area - 90% interval for the DSGE,
bold dashed line - IRF from the DSGE-VAR, dashed lines - 90% bands for the DSGE-VAR.